Sobolev multiplication below the borderline

Recall that from the Sobolev embedding theorem, we have continuous embeddings

$$L^p_k \hookrightarrow \begin{cases} C^0 & \text{if } k/n - 1/p > 0, \\ L^{\frac{1}{1/p-k/n}} & \text{if } k/n - 1/p < 0, \\ L^r \ \forall r & \text{if } k/n - 1/p = 0. \end{cases}$$

Whenever $k/n - 1/p \le 0$, the Sobolev space L^p_k contains discontinuous functions, and thus does not embed into C^0 or even L^∞.

Theorem 1. *Consider two Sobolev spaces L^p_k and L^q_ℓ such that* strict *inequality holds: $k/n - 1/p < 0$ and $\ell/n - 1/q < 0$. Then multiplication of functions extends to a continuous map of Sobolev spaces*

$$L^p_k \times L^q_\ell \to L^r_m$$

whenever $m \in \mathbb{Z}$ with $0 \le m \le \min(k, \ell)$, and r is such that

$$0 < m/n + (1/p - k/n) + (1/q - \ell/n) \le 1/r \le 1.$$

In other words, there exists a constant $C_{Xpqrk\ell m}$ (depending only on X and the Sobolev indices) such that

$$\|fg\|_{L^r_m} \le C_{Xpqrk\ell m} \|f\|_{L^p_k} \|g\|_{L^q_\ell}.$$

Remark. Originally we defined the Sobolev space L^r_m for $m \in \mathbb{R}$ and $r \in [1, \infty]$ via the spectral decomposition of the Laplacian:

$$L^r_m := \left\{ \text{distributions } f \ \middle| \ (1 + \Delta)^{m/2} f \in L^r \right\}.$$

But here we will assume without proof that for $m \in \mathbb{Z}_{\ge 0}$, the L^r_m norm is equivalent to

$$\|f\|_{L^r_m} \sim \sum_{i=0}^{m} \left\|\nabla^i f\right\|_{L^r}.$$

Proof. To prove our desired estimate, it suffices (by the iterated product rule and triangle inequality) to find an estimate

$$\left\|(\nabla^a f)(\nabla^b g)\right\|_{L^r} \le C_{Xpqrk\ell ab} \|f\|_{L^p_k} \|g\|_{L^q_\ell}$$

for each pair of integers $a, b \ge 0$ with $a + b \le m$.

We have continuous maps

$$L^p_k \xrightarrow{\nabla^a} L^p_{k-a} \hookrightarrow L^{\frac{1}{1/p-(k-a)/n}}$$

and

$$L^q_\ell \xrightarrow{\nabla^b} L^q_{\ell-b} \hookrightarrow L^{\frac{1}{1/q-(\ell-b)/n}}.$$

By Hölder's inequality

$$\|fg\|_{L^{1/(u+v)}} \le \|f\|_{L^{1/u}} \|g\|_{L^{1/v}},$$

so multiplication is continuous on

$$L^{1/u} \times L^{1/v} \to L^{1/(u+v)}.$$

(This assumes that $a+b \leq 1$ so that $L^{1/(a+b)}$ is still a Banach space.) Thus mutiplication is continuous on

$$L^{\frac{1}{1/p-(k-a)/n}} \times L^{\frac{1}{1/q-(\ell-b)/n}} \to L^{\frac{1}{-(m-a-b)/n+m/n+(1/p-k/n)+(1/q-\ell/n)}} \hookrightarrow L^r,$$

and composition of these maps gives a continuous map

$$L^p_k \times L^q_\ell \to L^r.$$

Our desired constant $C_{X^{pqr}_{k\ell ab}}$ is by definition the operator norm of this map. □

Sobolev multiplication above the borderline

Theorem 2. *Suppose that $k/n - 1/p > 0$ and $L^p_k \hookrightarrow L^q_\ell$ (i.e. $k \geq \ell$ and $k/n - 1/p \geq \ell/n - 1/q$). If $\ell \in \mathbb{Z}_{\geq 0}$, then multiplication of functions extends to a continuous map of Sobolev spaces*

$$L^p_k \times L^q_\ell \to L^q_\ell.$$

Proof. As in the previous section, we want to check

$$\left\|(\nabla^a f)(\nabla^b g)\right\|_{L^q} \leq C_{Xpqk\ell ab} \left\|f\right\|_{L^p_k} \left\|g\right\|_{L^q_\ell}$$

for all nonnegative integers a, b such that $a + b \leq \ell$. We have to deal with the cases

$$L^p_k \xrightarrow{\nabla^a} L^p_{k-a} \hookrightarrow \begin{cases} C^0 & \text{if } (k-a)/n - 1/p > 0, \\ L^{\frac{1}{1/p-(k-a)/n}} & \text{if } (k-a)/n - 1/p < 0, \\ L^r \ \forall r & \text{if } (k-a)/n - 1/p = 0, \end{cases}$$

and

$$L^q_\ell \xrightarrow{\nabla^b} L^q_{\ell-b} \hookrightarrow \begin{cases} C^0 & \text{if } (\ell-b)/n - 1/q > 0, \\ L^{\frac{1}{1/q-(\ell-b)/n}} & \text{if } (\ell-b)/n - 1/q < 0, \\ L^r \ \forall r & \text{if } (\ell-b)/n - 1/q = 0. \end{cases}$$

□

- If both $(k-a)/n - 1/p < 0$ and $(\ell-b)/n - 1/q < 0$, then as in the previous proof,

  $$L^{\frac{1}{1/p-(k-a)/n}} \times L^{\frac{1}{1/q-(\ell-b)/n}} \hookrightarrow L^q$$

  as desired since

  $$1/p - (k-a)/n + 1/q - (\ell-b)/n \overset{a+b\leq\ell}{\leq} 1/p - k/n + 1/q \overset{k/n-1/p>0}{<} 1/q.$$

- Suppose instead that $(\ell - b)/n - 1/q < 0$ and $(k-a)/n - 1/p \geq 0$. Then

$$L^q_{\ell-b} \hookrightarrow L^{\frac{1}{1/q-(\ell-b)/n}},$$

and

$$L^p_{k-a} \hookrightarrow C^0 \text{ or } L^r \ \forall r.$$

  - In the case $b = \ell$, we must have $a = 0$, so $(k-a)/n - 1/p = k/n - 1/p > 0$ and thus $L^p_{k-a} \hookrightarrow C^0$. Therefore,

$$L^p_{k-a} \times L^q_{\ell-b} \hookrightarrow C^0 \times L^{\frac{1}{1/q-(\ell-b)/n}} = C^0 \times L^q \to L^q.$$

  - If $b \neq \ell$, then $L^p_{k-a} \hookrightarrow L^r$ for any r, so

$$L^q_{\ell-b} \times L^p_{k-a} \to L^{\frac{1}{1/q-(\ell-b)/n+1/r}} = L^q \text{ for } r = n/(\ell-b).$$

- Suppose $(k-a)/n - 1/p < 0$ and $(\ell-b)/n - 1/q \geq 0$. Then

$$L^p_{k-a} \hookrightarrow L^{\frac{1}{1/p-(k-a)/n}} \hookrightarrow L^q$$

because

$$1/p - (k-a)/n \leq 1/p - (k-\ell)/n \leq 1/q.$$

For equality to hold, both $a = \ell$ and $k/n - 1/p = \ell/n - 1/q$, thus $b = 0$ and $(\ell - b)/n - 1/q = k/n - 1/p > 0$. Therefore, $L^q_{\ell-b} \hookrightarrow C^0$.

- Finally, suppose $(k-a)/n - 1/p \geq 0$ and $(\ell-b)/n - 1/q \geq 0$. Then choosing $r = 2q$, we obtain $L^p_{k-a} \hookrightarrow L^{2q}$ and $L^q_{\ell-b} \hookrightarrow L^{2q}$, so $L^p_{k-a} \times L^q_{\ell-b} \to L^q$.

Sobolev multiplication on the borderline

Theorem 3. *Suppose that $k/n - 1/p = 0$ and $L^p_k \hookrightarrow L^q_\ell$ (i.e. $k \geq \ell$ and $\ell/n - 1/q \leq 0$). If $\ell \in \mathbb{Z}_{\geq 0}$, then multiplication of functions extends to a continuous map of Sobolev spaces*

$$(L^p_k \cap L^\infty) \times (L^q_\ell \cap L^\infty) \to (L^q_\ell \cap L^\infty).$$

Furthermore, if the other function is below the borderline $\ell/n - 1/q < 0$, then we have the stronger result

$$(L^p_k \cap L^\infty) \times L^q_\ell \to L^q_\ell.$$

Proof. We will proceed by assuming $\ell/n - 1/q \leq 0$, proving continuity of the second multiplication, and realizing $\ell/n - 1/q = 0$ as an exceptional case.

Continuity of $(L^p_k \cap L^\infty) \times L^q_\ell \to L^q_\ell$ is equivalent to estimates of the form

$$\left\|(\nabla^a f)(\nabla^b g)\right\|_{L^q} \leq C_{Xpqabk\ell}\left(\|f\|_{L^p_k} + \|f\|_{L^\infty}\right)\|g\|_{L^q_\ell}$$

for all nonnegative integers a, b such that $a + b \le \ell$. For the case $a = 0$ we use the L^∞, and for the case $a > 0$ we use the Sobolev embedding theorem to obtain

$$(L^p_k \cap L^\infty) \xrightarrow{\nabla^a} L^p_{k-a} \hookrightarrow \begin{cases} L^\infty & \text{if } a = 0 \\ L^{\frac{1}{a/n}} & \text{if } a > 0 \end{cases} = L^{n/a}, \text{ where } n/0 := \infty.$$

In particular, for all a we have an estimate of the form

$$\|\nabla^a f\|_{L^{n/a}} \le C\left(\|f\|_{L^p_k} + \|f\|_{L^\infty}\right).$$

Note that the $\|f\|_{L^\infty}$ term on the right is essential to cover the case $a = 0$ since $L^p_k \not\hookrightarrow L^\infty$.

For the other factor,

$$L^q_\ell \xrightarrow{\nabla^b} L^q_{\ell-b} \hookrightarrow \begin{cases} L^r \ \forall r < \infty & \text{if } b = 0 \text{ and } \ell/n - 1/q = 0, \\ L^{\frac{1}{1/q-(\ell-b)/n}} & \text{else.} \end{cases}$$

In the latter case,

$$(L^p_k \cap L^\infty) \times L^q_\ell \hookrightarrow L^{\frac{1}{1/q-(\ell-a-b)/n}} \hookrightarrow L^q,$$

as desired. In the former case,

$$(L^p_k \cap L^\infty) \times L^q_\ell \hookrightarrow L^{\frac{1}{a/n+1/r}} \ \forall r < \infty.$$

Thus we can choose r large enough if $a/n < 1/q$. But

$$a/n \le \ell/n \le 1/q.$$

Thus we are safe only when we have strict inequality $\ell/n - 1/q < 0$. Otherwise we should take $g \in L^q_b \cap L^\infty$ so that we can effectively set $r = \infty$ and obtain $L^{\frac{1}{a/n+1/r}} \hookrightarrow L^q$. This proves continuity of

$$(L^p_k \cap L^\infty) \times (L^q_\ell \cap L^\infty) \to L^q_\ell$$

and

$$(L^p_k \cap L^\infty) \times L^q_\ell \to L^q_\ell \text{ when } \ell/n - 1/q < 0.$$

It remains to prove

$$(L^p_k \cap L^\infty) \times (L^q_\ell \cap L^\infty) \to L^\infty,$$

but this is obvious since

$$L^\infty \times L^\infty \to L^\infty.$$

□

Remark. These estimates can be generalized to non-integral Sobolev spaces by using interpolation theory.