At this point, the natural question is: given an intersection form, how many smooth 4-manifolds does it correspond to?

For all definite forms except for the diagonal, the answer is zero by Donaldson's theorem.

Next we list what we have: $Q_{\mathbb{CP}^2} = (+1)$, $Q_{\overline{\mathbb{CP}^2}} = (-1)$. $Q_{S^2\times S^2} = H$. These realize all intersection forms without a $\pm E_8$ factor.

Note that if Q_X is odd, then X is homeomorphic to $m\mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$.

To see examples of manifolds with an E_8 factor, we go to complex geometry. The family Calabi-Yau manifolds of complex dimension 2 is called the K3 surfaces. Since they are all diffeomorphic, differential topologists call them *the* K3 surface. One description is the quartic hypersurface in $\mathbb{CP}^3$ defined by

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, \quad (z_0 : z_1 : z_2 : z_3) \in \mathbb{CP}^3.$$

It turns out that

$$Q_{\mathrm{K3}} = -2E_8 \oplus 3H.$$

It's not possible to find a smooth 4-manifold with a single copy of E_8. For now, we state without proof that for X a simply connected closed 4-manifold,

$$Q_X \text{ even} \iff \text{tangent bundle of } X \text{admits a spin structure} =: X \text{ is spin}$$

There are several interesting theorems on spin manifolds.

Theorem (Rokhlin). *If X is a smooth spin 4-manifold, then the signature satisfies $\sigma(X) \equiv 0 \pmod{16}$.*

Since a closed simply-connected spin 4-manifold has intersection form

$$Q \sim \pm m\, E_8 \oplus n\, H,$$

we compute

$$\sigma(Q) = \pm 8m.$$

Thus Rokhlin's theorem implies m is even.

Next we ask whether it is possible to reduce the number of H in K3. Furuta used the Seiberg-Witten equations to prove

Theorem (Furuta (2001)). *If X is a closed oriented spin 4-manifold with $b_2(X) \neq 0$, then*

$$b_2(X) \geq \frac{10}{8}\,|\sigma(X)| + 2.$$

Substituting $b_2(X) = 8m + 2n$ and $|\sigma(X)| = 8m$, the above inequality is equivalent to $n \geq m + 1$. Thus for K3, $n \geq 3$, so we have the minimal number of H.

Closely related

Conjecture ($\frac{11}{8}$). *If X is a closed oriented spin 4-manifold, then*

$$b_2(X) \geq \frac{11}{8}\,|\sigma(X)|.$$

This is equivalent to $n \geq \frac{3}{2}m$. By Freedman's classification, this is equivalent to the conjecture that any simply-connected closed oriented spin 4-manifold be homeomorphic to

$$\tfrac{m}{2}\,\mathrm{K3}\,\#\,(n - \tfrac{3}{2}m)\,(S^2 \times S^2),$$

where of course the number of copies of each type is a nonnegative integer.

Assuming the $\frac{11}{8}$ conjecture, all smooth closed simply connected 4-manifolds are homeomorphic to connected sums of $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$, K3, $\overline{\mathrm{K3}}$, and $S^2 \times S^2 = \overline{S^2 \times S^2}$. Furthermore, based on the classification theorem, we can read off all the relations

$$\begin{aligned}
\mathrm{K3}\#\overline{\mathrm{K3}} &= 22\,S^2 \times S^2,\\
\mathrm{K3}\#\mathbb{CP}^2 &= 4\mathbb{CP}^2\#19\overline{\mathbb{CP}^2},\\
\mathrm{K3}\#\overline{\mathbb{CP}^2} &= 3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2},\\
\mathbb{CP}^2\#S^2 \times S^2 &= 2\mathbb{CP}^2\#\overline{\mathbb{CP}^2},
\end{aligned}$$

plus the corresponding identities obtained from the above by reversing the orientations. Thus the smooth classification problem is focused on classifying exotic structures on these connected sums.

Čech cohomology

It is extremely useful to be able to switch perspectives on cohomology. De Rham cohomology relates to calculus of differential forms. Singular cohomology relates to submanifolds. Čech cohomology will relate to fiber bundles. The equivalence of these theories provides deep connections between these subjects.

Suppose we have an open cover $\{U_\alpha\}$ of our manifold X. We define the chain complex $\check{C}^p(\{U_\alpha\};A)$ as follows. Denote multiple intersections by

$$U_{\alpha\beta} := U_\alpha \cap U_\beta, \quad U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma, \quad \text{etc.}$$

A Čech p-cochain ϕ associates to each $p+1$-fold intersection $U_{\alpha_0\alpha_1\cdots\alpha_p}$ a locally constant function $\phi_{\alpha_0\alpha_1\cdots\alpha_p} : U_{\alpha_0\alpha_1\cdots\alpha_p} \to A$.

$$\check{C}^p(\{U_\alpha\};A) := \left\{\phi = \left\{\phi_{\alpha_0\alpha_1\cdots\alpha_p} : U_{\alpha_0\alpha_1\cdots\alpha_p} \to A \text{ locally constant}\right\}\right\}.$$

The coboundary map is

$$\begin{aligned}
d : \check{C}^p(\{U_\alpha\};A) &\to \check{C}^{p+1}(\{U_\alpha\};A)\\
(d\phi)_{\alpha_0\alpha_1\cdots\alpha_{p+1}} &:= \sum_{k=0}^{p+1}(-1)^k\,\phi_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_{p+1}},
\end{aligned}$$

where $\widehat{\alpha_k}$ denotes omission of α_k. It's easy to verify that $d^2 = 0$. If $\phi \in \check{C}^0(\{U_\alpha\};A)$, then ϕ defines a collection of locally constant functions $\phi_\alpha : U_\alpha \to A$. If $d\phi = 0$, then for each $U_\alpha \cap U_\beta$, we have

$0 = \phi_\alpha - \phi_\beta$, so the ϕ_α agree on the overlaps and determine a locally constant function $\phi : X \to A$. As usual, we define

$$\check{H}^p(\{U_\alpha\}; A) := \frac{\ker d}{\operatorname{image} d}.$$

But we want cohomology to depend on X rather than a given cover. A different open cover $\{V_\beta\}_{\beta \in J}$ is called a *refinement* of $\{U_\alpha\}_{\alpha \in I}$ if each V_β is contained in some U_α. Fixing a choice $V_\beta \subset U_{\tau(\beta)}$ of some function $\tau : J \to I$ induces a restriction map

$$\check{C}^p(\{U_\alpha\}; A) \to \check{C}^p(\{V_\beta\}; A).$$

The induced map on cohomology $\check{H}^p(\{U_\alpha\}; A) \to \check{H}^p(\{V_\beta\}; A)$ does not depend on the choice of τ. Note that any two open covers $\{U_\alpha\}_{\alpha \in I}$ and $\{V_\beta\}_{\beta \in J}$ have a common refinement $\{U_\alpha \cap V_\beta\}_{(\alpha,\beta) \in I \times J}$. We define

$$\check{H}^p(X; A) := \operatorname*{dir\text{-}lim}_{\{U_\alpha\} \text{ open cover}} \check{H}^p(\{U_\alpha\}; A).$$

This means that any element of $\check{H}^p(X; A)$ is represented as a Čech cocycle with respect to some specific cover $\{U_\alpha\}$, and two elements in $\check{H}^p(X; A)$ are equal iff they become equal under a common refinement. Thankfully, we don't have to worry about this direct limit in practice.

A cover $\{U_\alpha\}$ is called a *good cover* if each U_α is contractible, as well as each finite intersection $U_{\alpha_0 \cdots \alpha_p}$. If $\{U_\alpha\}$ is a good cover, then $\check{H}^p(X; A) = \check{H}^p(\{U_\alpha\}; A)$.

Good covers always exist on a smooth manifold. We can pick a Riemannian metric, and then use metric balls which are sufficiently small to be *geodesically convex*. Any geodesically convex subset is contractible, and intersections of geodesically convex subsets are geodesically convex. (The naive strategy would be to use convex coordinate charts, however convexity is not preserved under coordinate change. But geodesic convexity with respect to a fixed Riemannian metric is.)

Like our previous cohomology theories, $\check{H}^p(X; A)$ is canonically isomorphic to sheaf cohomology, and thus can be identified with de Rham and singular cohomology.

Bundle theory

A *smooth Euclidean vector bundle of rank k over a manifold* X is a projection map $\pi : E \to X$ such that each fiber $\pi^{-1}(X)$ is a Euclidean vector space, and they fit together via smooth local trivializations

$$\begin{array}{ccc} E|_{U_\alpha} & \xleftarrow[\{\phi_\alpha\}]{\cong} & U_\alpha \times \mathbb{R}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & U_\alpha & \end{array},$$

where the $\{U_\alpha\}$ cover X, and the ϕ_α parameterize fiberwise isometries $\phi_\alpha(x) : \mathbb{R}^k \to E|_x$. Note that $\phi = \{\phi_\alpha\}$ defines a map

$$\begin{array}{ccc} E & \xleftarrow{\ \phi\ } & \coprod_\alpha U_\alpha \times \mathbb{R}^k \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & X & \end{array}$$

which covers E, and two points $(x, v)_\alpha \in U_\alpha \times \mathbb{R}^k$ and $(x, w)_\beta \in U_\alpha \times \mathbb{R}^k$ map to the same point in E iff

$$\phi_\alpha(x)v = \phi_\beta(x)w \iff v = \phi_\alpha^{-1}(x)\phi_\beta(x)w = \phi_{\alpha\beta}(x)w,$$

where $\phi_{\alpha\beta}(x) := \phi_\alpha^{-1}(x)\phi_\beta(x) \in \mathrm{Isom}(\mathbb{R}^k \leftarrow \mathbb{R}^k) =: \mathrm{O}(k)$. As $\phi_{\alpha\beta}$ ranges over all possible x, it defines a function $\phi_{\alpha\beta} \in C^\infty(U_{\alpha\beta}; \mathrm{O}(k))$ called a *transition function*. After identifying corresponding points $\left[x, \phi_{\alpha\beta}(x)v\right]_\alpha \sim [x, v]_\beta$, the map ϕ induces an isometry of Euclidean vector bundles

$$\begin{array}{ccc} E & \xleftarrow[\phi]{\cong} & \coprod_\alpha U_\alpha \times \mathbb{R}^k / \sim . \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & X & \end{array}$$

Thus every smooth Euclidean vector bundle is isomorphic (=isometric) to a vector bundle determined by transition functions $\phi_{\alpha\beta} \in C^\infty(U_{\alpha\beta}; \mathrm{O}(k))$.