Notes on classification

Classification of 3-manifolds is based on Thurston's geometrization conjecture. Roughly, this states that every three-manifold can be decomposed in terms of certain "geometric" pieces. Perelman showed that these pieces can be obtained via the Ricci flow

$$\partial_t g_{ij} = -2R_{ij},$$

where g_{ij} is a Riemannian metric, and R_{ij} is the Ricci curvature tensor. This is an evolution equation which behaves like a heat equation, tending to uniformize the curvature. Singularities develop, for instance, as the various geometric pieces pinch off, and one of the major technical obstacles is understanding how to deal with these singularities so that the flow can continue. Perelman's recent results essentially reduce the classification problem to understanding the geometric pieces and their possible quotients. Consequently, the theory of 3-manifolds involves much group theory related to the possible fundamental groups which arise.

In higher dimensions, the group theory becomes literally impossible. Any finitely presented group can appear as $\pi_1(X)$ for compact X when $\dim X \geq 4$. The classification of finitely presented groups is undecidable. Philisophically, the idea is that given any fixed axiom system, it's possible to manufacture a group presentation which effectively encodes a statement such as, "triviality of this group is equivalent to a proof with your axioms that this group is nontrivial." Assuming consistency of your axioms, such a group must be nontrivial. However, your axioms cannot provide a proof.

Since the general classification problem is doomed from the start, typically one focuses on classifying simply connected manifolds in these dimensions. When $n \geq 5$, classification of smooth manifolds is generally considered well-understood due to surgery theory, which essentially reduces classification to an algebraic problem. However, in dimension 4, things go wrong due to failure of the "Whitney trick." Given two submanifolds P and Q of complementary dimension, they can be perturbed to intersect transversely to meet in finitely many points. If everything is oriented, then these intersection points have signs. One wants to be able to cancel intersection points which have opposite signs. The strategy is to form a loop by taking a path inside each of P and Q between the intersection points. We wish to to fill this in with a smoothly embedded "Whitney disk," which then allows us to slide apart the surfaces. In dimension 4, such disks will generally have self-intersections. Roughly speaking, Freedman's classification of simply connected topological 4-manifolds uses an infinite sequence of modifications, called *Casson handles*, to eliminate self-intersections, but not smoothly.

General coefficients for cohomology

There are no fractions involved in the definition of singular cohomology. Thus it makes sense to replace all instances of $\mathbb{R}$ by $\mathbb{Z}$ in the definition of singular cohomology. This version resolves the sheaf of locally constant $\mathbb{Z}$-valued functions, and is more delicate and powerful than its $\mathbb{R}$ counterpart. More generally, this construction makes sense over any abelian group A, which we denote with cochain groups $C^\bullet(X; A)$ and cohomology groups $H^\bullet(X; A)$. There is a cup product structure whenever A is a ring R. Most common are $R \in \{\mathbb{R}, \mathbb{Z}, \mathbb{Z}_2\}$.

Sadly, de Rham theory is capable only of computing $H^\bullet(X;\mathbb{R})$.

There is a "universal coefficient theorem" which computes cohomology with general coefficients, but first we need homology.

Singular homology

We observe that $C^p(X;A)$ is the dual space

$$C^p(X;A) = C_p(X;A)^* := \mathrm{Hom}(C_p(X;A), A),$$

where $C_p(X;A)$ is the vector space with a basis element corresponding to each $\sigma \in \Delta_p(X)$. In other words, finite formal linear combinations

$$C_p(X;A) := \sum_{\sigma\in\Delta_p(X)} c_\sigma\,\sigma,\ \text{where finitely many } c_\sigma \in A \text{ are nonzero.}$$

Furthermore, there is a linear differential $\partial : C_p(X;A) \to C_{p-1}(X;A)$ determined by

$$\partial\sigma := \sum_{i=0}^{p} (-1)^i F_i \sigma,$$

satisfying $\partial^2 = 0$. This fits into a sequence

$$\cdots \xrightarrow{\partial} C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X),$$

where we leave A implicity for brevity. The dual of this sequence is precisely the sequence

$$C^0(X) \xrightarrow{d} C^1(X) \xrightarrow{d} C^2(X) \xrightarrow{d} \cdots$$

of singular cohomology. We define singular homology

$$H_p(X) := \frac{\ker\left(C_p(X) \to C_{p-1}(X)\right)}{\mathrm{image}\left(C_{p+1}(X) \to C_p(X)\right)}.$$

For example, $H_0(X;A) \cong A^{\#\text{components}(X)}$. Also, if X is connected, then $H_1(X;\mathbb{Z}) \cong \pi_1^{\mathrm{ab}}(X, x_0)$, where π_1^{ab} denotes the abelianization of the fundamental group. In particular, for the Poincaré homology sphere P, $H_1(P;\mathbb{Z}) = 0$.

Universal coefficients and Poincaré duality

Cohomology is dual to homology in two distinct ways: universal coefficients and Poincaré duality.

The more straightforward is the universal coefficient theorem. One might hope that since cochains are dual to chains, maybe cohomology is dual to homology. This is almost true, but not quite. The situation is described by the split exact sequence

$$0 \to \mathrm{Ext}(H_{i-1}(X;\mathbb{Z}), A) \to H^i(X;A) \to \mathrm{Hom}(H_i(X;\mathbb{Z}), A) \to 0.$$

Whenever we have a *short exact sequence* of abelian groups $0 \to A \to B \to C \to 0$, it follows that A can be identified with its image $A \subset B$ and $C \cong B/A$. (These are straightforward consequences of the definition of exactness.) The term *split* means that there is a subgroup of B representing the quotient group C, so that $B \cong A \oplus C$. (However, this splitting it rarely unique.)

To be more concrete regarding the universal coefficient theorem, suppose that X is closed. Then $H_i(X;\mathbb{Z})$ is a finitely generated abelian group. According to the classification theorem for finitely generated abelian groups, there is a split exact sequence

$$0 \to T_i(X) \to H_i(X;\mathbb{Z}) \to H_i^{\text{free}}(X) \to 0,$$

where the *torsion subgroup* $T_i(X) \subset H_i(X;\mathbb{Z})$ is the finite subgroup consisting of elements of finite order. Furthermore, $T_i(X)$ is isomorphic to a direct sum of finite cyclic groups $T_i(X) \cong \mathbb{Z}_{i_1} \oplus \cdots \oplus \mathbb{Z}_{i_k}$. The quotient group $H_i(X;\mathbb{Z})/T_i(X)$ is a free group denoted $H_i^{\text{free}}(X)$. Since the sequence is split, there exists isomorphisms

$$H_i(X;\mathbb{Z}) \cong H_i^{\text{free}}(X) \oplus T_i(X).$$

Upon choosing a basis for $H_i^{\text{free}}(X)$, we obtain an isomorphism

$$H_i(X;\mathbb{Z}) \cong \mathbb{Z}^{b_i(X)} \oplus T_i(X),$$

for some nonnegative integer $b_i(X)$.

To be continued...