Topological chirality

We can extend our diagram from last time:

orientation-preserving diffeomorphic	$\Rightarrow$	orientation-preserving homeomorphic	$\Rightarrow$	positively homotopy equivalent		
$\Downarrow$		$\Downarrow$		$\Downarrow$		
diffeomorphic	$\Rightarrow$	homeomorphic	$\Rightarrow$	homotopy equivalent	$\Rightarrow$	isomorphic cohomology

and all these equivalences are induced by functors.

It can happen that a manifold is smoothly chiral, but topologically achiral (admitting an orientation-reversing homeomorphism). Several exotic spheres provide examples. Oriented exotic n-spheres form an abelian monoid (group without inverses) under connected sum. When $n \neq 4$, there is an inverse is given by orientation reversal, making it an abelian group Θ_n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Θ_n	0	0	0	?	0	0	$\mathbb{Z}_{28}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_6$	$\mathbb{Z}_{992}$	0	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_{8128} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$

It is unknown whether exotic S^4 exist.

Consider the exotic 7-spheres group $\mathbb{Z}_{28}$. Recall that all $\{S_i^7\}_{i\in\mathbb{Z}_{28}}$ are homeomorphic to the standard S_0^7, and $S_0^7 \overset{\text{homeo}}{\cong} \overline{S_0^7}$ via reflection, so each S_i^7 admits an orientation-reversing homeomorphism. However, the $\{S_i^7\}_{i\in\mathbb{Z}_{28}}$ are distinct as smooth oriented manifolds. Thus

$$S_i^7 \overset{\text{diffeo}}{\cong} \overline{S_i^7} = S_{-i}^7 \iff i \equiv -i \pmod{\{28\}} \iff i \in \{0, 14\}.$$

Thus exactly 26 of the exotic 7-spheres are chiral.

Homotopy

Definition. A (smooth) homotopy between two maps $f_i : X \to Y$, $i \in \{0,1\}$ is a (smooth) continuous map $f : X \times [0,1] \to Y$ such that $f_i = f|_{X\times\{i\}}$.

Both homotopy and smooth homotopy are equivalence relations. Between two smooth manifolds, one could choose between either smooth homotopy or regular continuous homotopy. Conveniently, these notions are essentially equivalent.

Theorem (Smooth approximation). *Any continuous map $f : X \to Y$ between smooth manifolds is homotopic to a smooth map. Moreover, if f_0 and f_1 are homotopic, then they are smoothly homotopic.*

Thus homotopy classes of maps between smooth manifolds are equivalent, regardless of whether or not the maps are required to be smooth.

In the homotopy category of continuous maps modulo homotopy, let's examine the notion of isomorphism, which is called *homotopy equivalence*. The morphism represented by $f : X \to Y$ is an isomorphism if there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. For example, the inclusion of $f : S^1 \hookrightarrow \mathbb{R}^2 - \{0\}$ is a homotopy equivalence since $g : \mathbb{R}^2 - \{0\} \to S^1$ defined by $x \mapsto x/|x|$ satisfies $g \circ f = \mathrm{Id}_{S^1}$, and $f \circ g : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$ by $x \mapsto x/|x|$ is homotopic to the identity by

$$x \mapsto \frac{x}{t + (1-t)|x|}.$$

As in this example, a noncompact manifold can be homotopy equivalent to a manifold of a lower dimension. Any contractible manifold is homotopy equivalent to a point.

Fundamental group

Consider a manifold X with a specified point $x_0 \in X$. Define $\pi_1(X, x_0)$ to be the set of homotopy classes of loops starting and ending at x_0, where any homotopy is also required to fix the endpoints at x_0. It's routine to check that $\pi_1(X, x_0)$ is a group, where composition corresponds to concatenation of loops, and reversal of a path gives its inverse. When X is connected, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic with an isomorphism induced from choice of a path connecting x_0 to x_1. However, the isomorphism depends on the homotopy class of such a path. The group $\pi_1(X, x_0)$ is called the *fundamental group* of X, and it is a homotopy invariant of X. (A homotopy equivalence of manifolds induces an isomorphism of fundamental groups.)

If X is connected and $\pi_1(X, x_0) = 0$, then X is said to be *simply connected*. For example, S^n is simply connected for $n \geq 2$.

In principle, this would be a good time to discuss covering spaces, but in the interest of time, we will skip them for now. Most manifolds we consider will be simply connected anyway. Instead, I leave you with a theorem which should suffice for most of our purposes.

Theorem. *If G is a group acting freely and properly discontinuously on a simply connected manifold X, then X/G is a manifold with $\pi_1(X/G, [x_0]) = G$.*

Poincaré homology sphere

The symmetries of the icosahedron form a subgroup $I \subset \mathrm{SO}(3)$ of order 60 called the icosahedral group. The Poincaré homology sphere is the quotient space $P = \mathrm{SO}(3)/I$. Topologically, the group $\mathrm{SO}(3)$ is $\mathbb{RP}^3 = S^3/\text{antipodal map}$. (We will explain this in a moment.) There is a group $\tilde{I}$ of order 120 called the binary icosahedral group such that $P = \mathrm{SO}(3)/I = S^3/\tilde{I}$. Since S^3 is simply connected, it follows from the above theorem that $\pi_1(P) = \tilde{I}$. The group $\tilde{I}$ is *perfect*, meaning that it is generated by its commutators. Consequently, the abelianization of $\tilde{I}$ (the group obtained by imposing commutativity) is trivial. Once we understand more about homology and cohomology, we will see how this implies that the cohomology ring of P is isomorphic to that of S^3. That's why P is called a (co)homology sphere.

Lens spaces

Let p and q be coprime integers. The lens space $L(p;q)$ is the quotient space of the unit sphere $S^3 \subset \mathbb{C}^2$ under the $\mathbb{Z}_p$ action generated by

$$(z_1, z_2) \sim (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2).$$

Thus $\pi_1(L(p;q), x_0) = \mathbb{Z}_p$.

Lens spaces provide many useful examples for understanding different levels of classification. For example, two lens spaces $L(p;q_1)$ and $L(p;q_2)$ are homeomorphic iff $q_1 \equiv \pm q_2^{\pm 1} \pmod{p}$, with all four possible choices of sign. More generally, they are homotopy equivalent iff $q_1 q_2 \equiv \pm n^2 \pmod{p}$ for some integer n.

Lens spaces also provide useful examples of chirality. For instance, while $L(5;2)$ is smoothly achiral, $L(5;1)$ is smoothly chiral but homotopically achiral. (There is a homotopy equivalence of $L(5;1)$ with itself, which flips orientation in the sense that the induced map on $H^3(L(5;1)) \cong \mathbb{R}$ is negative.)

Singular cohomology

We are long overdue to introduce singular cohomology. Recall that de Rham cohomology arose from (locally) resolving the locally constant functions via the de Rham complex

$$\Omega^0(X) \to \Omega^1(X) \to \cdots.$$

We will introduce an alternative resolution

$$C^0(X) \to C^1(X) \to \cdots,$$

where $C^p(X)$ will be the space of singular cochains (not to be confused with the space of functions with continuous derivatives of order p).

Before defining singular cochains $C^p(X)$, let's review the general setup of cohomology, and the direction in which we're going. There is an abstract construction called "sheaf cohomology." This can be realized in numerous ways, based on resolution of locally constant functions. Roughly speaking, the recipe is to express locally constant functions as the kernel of some linear map $d : C^0(X) \to C^1(X)$. There are many ways to go about this, but there are some technical conditions on the vector spaces involved. The chosen linear map will generally not be surjective, thus its image defines a subspace of $C^1(X)$. We want d to capture the *local* relations satisfied by the image of d, so we restrict to an open subset $U \subset X$ homeomorphic to $\mathbb{R}^n$. There we describe systematically the image of d in $C^1(U)$ by the kernel of another map also denoted by $d : C^1(U) \to C^2(U)$. By iterating this construction, we get an exact sequence describing higher-order relations satisfied by locally constant functions. Passing from $C^\bullet(U)$ to $C^\bullet(X)$, i.e. from local to global, the sequence need no longer be exact. The defect in exactness is measured by the quotient spaces $H^p(X)$. These should be considered as higher-order relations on locally constant functions which are *global* in nature. If this construction is done in such a way that the vector spaces behave nicely under restriction (i.e. there should be partitions of unity, or some other method of subdivision), then the spaces $H^p(X)$ are canonically isomorphic to sheaf cohomology. While de Rham cohomology and singular cohomology will seem completely different, they compute the same sheaf cohomology and are thus isomorphic. An excellent source for this topic is [War83].

Being locally path-connected, the topological components of a manifold coincide with the path components. Thus a function is locally constant iff it is constant along paths. Let $C^0(X)$ denote the space of arbitrary (not necessarily continuous) functions $f : X \to \mathbb{R}$. Note that $f \in C^0(X)$ is locally constant iff for every continuous path $\gamma : [0,1] \to \mathbb{R}$ we have $f(\gamma(1)) - f(\gamma(0)) = 0$. We wish to cast this condition as a linear map. Let Γ denote the set of continuous paths in X

$$\Gamma(X) := \left\{ \gamma : [0,1] \to X \;\middle|\; \gamma \text{ is continuous} \right\}.$$

Let $C^1(X)$ denote the vector space of arbitrary functions

$$C^1(X) := \{ g : \Gamma(X) \to \mathbb{R} \}.$$

Define $d : C^0(X) \to C^1(X)$ by

$$\underbrace{f}_{C^0(X)} \mapsto \underbrace{\big(\underbrace{\gamma}_{\Gamma(X)} \mapsto \underbrace{f(\gamma(1)) - f(\gamma(0))}_{\mathbb{R}} \big)}_{C^1(X)}.$$

Thus f is locally constant iff $df = 0$.

The next step in our program of resolving locally constant functions is to characterize (locally) the subspace $\text{image}(d) \subset C^1(X)$. For instance, $g = df$ satisfies the property that for every triangle $(\gamma_1, \gamma_2, \gamma_3)$ with paths γ_1 from a to b, γ_2 from b to c, and γ_3 from a to c, we have

$$\begin{aligned} & g(\gamma_1) + g(\gamma_2) - g(\gamma_3) \\ =& f(\gamma_1(1)) - f(\gamma_1(0)) + f(\gamma_2(1)) - f(\gamma_2(0)) - f(\gamma_3(1)) + f(\gamma_3(0)) \\ =& f(b) - f(a) + f(c) - f(b) - f(c) + f(a) = 0. \end{aligned}$$

Conversely, suppose that $g \in C^1(X)$ satisfies for all triangles $g(\gamma_1) + g(\gamma_2) - g(\gamma_3) = 0$. We wish to define $f(b) := g(\gamma_1)$ for any path from a to b, and show that $g = df$. First note that a constant path γ defines a trivial triangle (γ, γ, γ), thus $g(\gamma) + g(\gamma) - g(\gamma) = 0$, so $g(\gamma) = 0$ for any constant path. To see that f is well-defined, consider an alternate path $\tilde{\gamma}_1$ from a to b. Then $(\gamma_1, b, \tilde{\gamma}_1)$ is a triangle, so $g(\gamma_1) + 0 - g(\tilde{\gamma}_1) = 0$, thus f is well-defined. Similarly, it's straightforward to show that $g = df$. Thus this condition on triangles exactly characterizes $\text{image}(d) \subset C^1(X)$. However, these triangles do not suit our purposes for defining a well-behaved resolution. Homologically, we need the capacity to break things up into small pieces. For example, with differential forms we have partitions of unity. In order to resolve in terms of triangles, we need to be able to break a triangle into smaller triangles. For instance, if X is a circle, and a our triangle wraps around the circle, then we are stuck.

The way out is to consider only *solid* triangles. In this case, we can break up any triangle via barycentric subdivision. Locally, considering solid triangles is no extra restriction, since if our triangle lies inside some convex set, we can fill it in. Thus the proper condition is to require g to vanish along the boundary of a solid triangle.

The standard p-simplex is defined to be $\Delta_p := \left\{(x_0, \ldots, x_p) \;\middle|\; x_i \geq 0 \text{ for each } i, \text{ and } \sum x_i = 1\right\}$. This is the proper n-dimensional generalization of a point, line segment, triangle, tetrahedron, etc. The p-simplex has $p+1$ faces $\{F_i\}_{i=0}^{p}$ which are standard $p-1$-simplices determined by setting $x_i = 0$.

Define the set of simplices in X by $\Delta_p(X) := \left\{\sigma : \Delta_p \to X \;\middle|\; \sigma \text{ is continuous}\right\}$. (For example, $\Delta_0(X) \equiv X$ as sets.) Then $C^p(X)$ is the vector space

$$C^p(X) := \left\{h : \Delta_p(X) \to \mathbb{R}\right\}.$$

We define the map $d : C^p(X) \to C^{p+1}(X)$ by specifying the values of dh for any $\sigma \in \Delta_{p+1}(X)$:

$$dh(\sigma) := \sum_{i=0}^{p+1} (-1)^i h(F_i \sigma).$$

It's straightforward to show that $d^2 = 0$. This provides the desired resolution of locally constant functions by singular cochains

$$C^0(X) \to C^1(X) \to \cdots,$$

and once again,

$$H^p(X) := \frac{\ker\left(C^p(X) \to C^{p+1}(X)\right)}{\text{image}\left(C^{p-1}(X) \to C^p(X)\right)}.$$

For concreteness, let's work through the definition of $d : C^0(X) \to C^1(X)$.

For $C^0(X)$, note that $\Delta_0 = \{\text{pt}\}$, so $\Delta_0(X) = \{\sigma : \{\text{pt}\} \to X\}$, so $\Delta_0(X)$ is simply the set of points in X. Cochains $C^0(X) = \{h : \Delta_0(X) \to \mathbb{R}\}$ are arbitrary functions on X, not necessarily continuous.

For $C^1(X)$, note that $\Delta_1 \cong [0, 1]$, so $\Delta_1(X)$ is the set of paths σ in X which are parameterized along $[0, 1]$. Thus $C^0(X)$ is the set of functions which assign a real number to each such path. $F_0(\Delta_1)$ and $F_1(\Delta_1)$ correspond to the two endpoints of $[0, 1]$, say $\{1\}$ an $\{0\}$ respectively.

Finally, if $f \in C^0(X)$, then $df \in C^1(X)$ is given on $\sigma \in \Delta_1(X)$ by $df(\sigma) = f(F_0\sigma) - f(F_1\sigma) = f(\sigma(1)) - f(\sigma(0))$.

As with de Rham cohomlogy, we can define the cup product, but it's slightly more complicated. We don't need to explicitly distinguish between singular and de Rham cohomology, since they are naturally isomorphic as different resolutions of the locally constant sheaf.

Bibliography

[War83] Warner, F.W. *Foundations of differentiable manifolds and Lie groups*