Orientations, again

In dimension zero, there is a technical problem with my current definition of orientation. Before, I said that an orientation is determined by a positive atlas, i.e. the transition functions have positive Jacobian. A better (and otherwise equivalent) definition is to say that an orientation on an n-manifold X is determined by a nowhere vanishing n-form $\omega \in \Omega^n(X)$. Two n-forms determine the same orientation if they differ by multiplication by a positive function. Let's review the proof of equivalence of these two definitions.

Suppose we have a nowhere vanishing n-form ω, and we wish to construct a positive atlas. Without loss of generality, assume the domain of each coordinate chart is connected. In any chart ϕ_α with x-coordinates, $\omega = f_\alpha(x)\, dx^1 \wedge \cdots \wedge dx^n$ with either $f_\alpha > 0$ or $f_\alpha < 0$. After reflecting a coordinate in any charts where $f_\alpha < 0$, we get an atlas with $f_\alpha > 0$. The transformation rule for ω from a chart ϕ_β with y-coordinates is

$$\omega = f_\beta(y)\, dy^1 \wedge \cdots \wedge dy^n = f_\beta(x)\, \det \frac{\partial y^i}{\partial x^j}\, dx^1 \wedge \cdots \wedge dx^n = f_\alpha(x)\, dx^1 \wedge \cdots \wedge dx^n,$$

so $\det \frac{\partial y^i}{\partial x^j} = \frac{f_\beta}{f_\alpha} > 0$. Thus our atlas is positive.

To go the other way, suppose we have a positive atlas, and let ρ_α be a partition of unity subordinate to the atlas. Take $\omega = \sum_\alpha \rho_\alpha\, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$. A priori, each term belongs only to $\Omega^n(U_\alpha)$. But since ρ_α is supported in U_α, it extends by zero to all of X. Positivity of the atlas ensures that the terms never cancel. Thus we get a nowhere vanishing n-form.

For a 0-manifold, an orientation corresponds to assignment of + or − to each point.

If X is an oriented n-manifold, possibly with boundary, then it makes sense to integrate $\omega \in \Omega^n(X)$ if it is compactly supported, so we get a natural map

$$\Omega_c^n(X) \xrightarrow{\int} \mathbb{R}, \qquad \omega \mapsto \int_X \omega,$$

defined locally in a positive chart by

$$\int_{\mathbb{R}^n} f(x)\, dx^1 \wedge \cdots \wedge dx^n := \int_{\mathbb{R}^n} f(x)\, dx^1 \cdots dx^n.$$

Positivity of the atlas ensures that the sign of $f(x)$ is consistent. Stokes' Theorem states that if X is an oriented n-manifold, possibly with boundary, then for any $\eta \in \Omega_c^{n-1}(X)$ with compact support,

$$\int_{\partial X} \eta = \int_X d\eta.$$

If X is compact, oriented, and without boundary, then $H^n(X) \cong \mathbb{R}$, with a well-defined isomorphism

$$[\omega] \mapsto \int_X \omega.$$

Functorial view of classification

In dimension 2, we saw that cohomological classification coincides with topological classification (which also corresponds to smooth classification).

As we move into dimension 3, although smooth and topological classifications still coincide, we will encounter other levels of classification which do not. Namely,

$$\text{smooth oriented} \subset \text{smooth} \subset \text{topological} \subset \text{homotopy} \subset \text{cohomology}.$$

For example, the Poincare homology sphere $\mathrm{SO}(3)/I$, where I is the icosahedral group of order 60 has the same cohomology as S^3, but is not homotopy equivalent to it.

To be specific, there are functors between categories

$$\left\{ \begin{array}{c} \text{connected oriented smooth manifolds} \\ \text{orientation-preserving diffeomorphisms} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{connected smooth manifolds,} \\ \text{smooth maps} \end{array} \right\}$$
$$\left\{ \begin{array}{c} \text{connected smooth manifolds,} \\ \text{smooth maps} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{connected topological manifolds,} \\ \text{continuous maps} \end{array} \right\}$$
$$\left\{ \begin{array}{c} \text{connected topological manifolds,} \\ \text{continuous maps} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{connected topological manifolds,} \\ \text{homotopy classes of continuous maps} \end{array} \right\}$$
$$\left\{ \begin{array}{c} \text{connected topological manifolds,} \\ \text{homotopy classes of continuous maps} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{c} \text{graded-commutative rings,} \\ \text{degree-preserving homomorphisms} \end{array} \right\}$$

Functors are morphisms of categories. They send objects of one category to objects of another, and similarly for morphisms. They preserve identity morphisms and composition. The main consequence is that functors preserve isomorphisms. Suppose X_1 and X_2 are isomorphic in some category $\mathcal{C}$, i.e. there are $f : X_1 \to X_2$ and $g : X_2 \to X_1$ such that $g \circ f = \mathrm{Id}_{X_1}$ and $f \circ g = \mathrm{Id}_{X_2}$. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor, so that $Y_1 = F(X_1)$ and $Y_2 = F(X_2)$. Then there are morphisms $F(f) : Y_1 \to Y_2$ and $F(g) : Y_2 \to Y_1$ so that $F(g) \circ F(f) = F(g \circ f) = F(\mathrm{Id}_{X_1}) = \mathrm{Id}_{Y_1}$, and similarly $F(f) \circ F(g) = \mathrm{Id}_{Y_2}$. Thus isomorphic objects will remain isomorphic under a functor. But non-isomorphic objects might become isomorphic in the image of a functor.

Chirality

The easiest step to visualize is going between smooth manifolds and oriented smooth manifolds. In dimension 2, we can list all smooth manifolds. To understand oriented surfaces, we should first cross out the non-orientable surfaces. Now each orientable surface has two orientations. But these two orientations are isomorphic via a reflection. Thus each smooth orientable surface corresponds to exactly one smooth surface.

The situation in dimension 1 is similar, but dimension 0 is quite different. The + point and − point are not orientation-preserving diffeomorphic, despite being diffeomorphic. We call such manifolds *chiral*, meaning that they are not equivalent to their mirror image.

The next simplest example of a chiral manifold appears in dimension 4 for

$$\mathbb{CP}^2 := \frac{(\mathbb{C}^3 - \vec{0})}{(z_1 : z_2 : z_3) \sim (\lambda z_1 : \lambda z_2 : \lambda z_3)}.$$

This is a compact complex manifold. Every complex manifold has a natural orientation. One can compute that the Betti numbers are

i	0	1	2	3	4
b^i	1	0	1	0	1

Since $n/2 = 2$ is even, the intersection form is symmetric of rank 1. There are only two possibilities: either $(+1)$ or (-1). (Concretely, pick any generator $h \in H^2(\mathbb{CP}^2)$. By Poincaré duality, $0 \neq h^2 \in H^4(\mathbb{CP}^2)$. After replacing $h \mapsto h/\sqrt{\left|\int_{\mathbb{CP}^2} h^2\right|}$, we have $\int_{\mathbb{CP}^2} h^2 = \pm 1$.) A computation shows that for $\mathbb{CP}^2$ the intersection form is $(+1)$, i.e. the intersection form is positive-definite. To deduce chirality of $\mathbb{CP}^2$, we need functoriality of cohomology.

Differential forms and cohomology are functorial. Given a map between two manifolds $y : X \to Y$ (not necessarily of the same dimension, and not necessarily a diffeomorphism), and $\omega \in \Omega^p(Y)$ with $\omega = \sum_I f_I(y) dy^{I_1} \wedge \cdots \wedge dy^{I_p}$, we define the differential form $y^*(\omega) \in \Omega^p(X)$ locally by $y^*(\omega) = \sum_I f_I(y(x)) dy^{I_1}(x) \wedge \cdots \wedge dy^{I_p}(x)$ and expanding in terms of the x variables. This gives a map $y^* : \Omega^p(Y) \to \Omega^p(X)$. Although it goes in the opposite direction, it behaves as expected with composition. It's easy to check that this induces a ring homomorphism $y^* : H^\bullet(Y) \to H^\bullet(X)$, i.e. it respects the cup product structure.

Suppose $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2$ is any smooth map. Since $H^2(\mathbb{CP}^2)$ is one-dimensional and spanned by h, we have $\psi^* : H^2(\mathbb{CP}^2) \to H^2(\mathbb{CP}^2)$ is determined by $\psi^*(h) = \lambda h$ for some $\lambda \in \mathbb{R}$. Since $\psi^* : H^\bullet(\mathbb{CP}^2) \to H^\bullet(\mathbb{CP}^2)$ is a ring homomorphism,

$$\psi^*(h^2) = \psi^*(h)^2 = (\lambda h)^2 = \lambda^2 h^2.$$

It follows that $\int_{\mathbb{CP}^2} \psi^*(h^2) = \lambda^2 \geq 0$.

Suppose by contradiction that $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2$ is an orientation-reversing diffeomorphism. We can compute $\int_{\mathbb{CP}^2} \psi^*(h^2)$ by representing $h^2 = [\omega]$ and computing $\int_{\mathbb{CP}^2} \psi^*(\omega)$ where ψ is viewed as a change of coordinates. Thus the integral is the same, except for a change in sign due to the orientation reversal.

$$\int_{\mathbb{CP}^2} \psi^*(h^2) = \int_{\mathbb{CP}^2} \psi^*(\omega) = -\int_{\mathbb{CP}^2} \omega = -\int_{\mathbb{CP}^2} h^2 = -1.$$

Thus, as oriented manifolds, $\mathbb{CP}^2$ and the same underlying smooth manifold with the reversed orientation $\overline{\mathbb{CP}^2}$ are not oriented-diffeomorphic.

Warning The notation $\overline{\mathbb{CP}^2}$ has nothing to do with the conjugate complex structure. Since the complex dimension is even, the conjugate complex structure induces the same orientation. In fact, the conjugation map $(z_1 : z_2 : z_3) \mapsto (\overline{z_1} : \overline{z_2} : \overline{z_3})$ is well-defined, and orientation-preserving.

A more sophisticated way of stating the chirality proof is that the intersection form is an invariant of oriented diffeomorphism. Orientation reversal flips the sign of the intersection form. Since the intersection forms $(+1)$ of $\mathbb{CP}^2$ and (-1) of $\overline{\mathbb{CP}^2}$ are inequivalent as bilinear forms, $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ cannot be oriented-diffeomorphic.

More generally, X is compact, oriented, and dimension n with $n/2$ is even, then the intersection form Q is symmetric, and

$$b^2(X) \text{ odd} \implies \sigma(X) \neq 0 \iff Q \not\simeq -Q \implies X \text{ chiral}.$$

(The $\iff$ in the middle should not be obvious, but happens to be a simple consequence of classification of unimodular symmetric bilinear forms.)

Gauge theory is sensitive to orientation. The moduli spaces for $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$ look completely different. Thus we will consider them distinct manifolds.