Intro to cohomology, continued

Last time we saw the notion of an *exact sequence*, and the particular example:

$$\cdots \to \mathbb{R}^0 \longrightarrow \mathbb{R}^1 \xrightarrow{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} -2 & 1 & 1 \end{pmatrix}} \mathbb{R}^1 \longrightarrow \mathbb{R}^0 \longrightarrow \mathbb{R}^0 \to \cdots.$$

The defining property is that the kernel of any linear map equals the image of the previous. The rows of each matrix describe the dependencies among the rows of the previous matrix. The idea is that either side

$$\cdots \to \mathbb{R}^0 \longrightarrow \mathbb{R}^1 \xrightarrow{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}} \mathbb{R}^3$$

or

$$\mathbb{R}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}} \mathbb{R}^3 \xrightarrow{\begin{pmatrix} -2 & 1 & 1 \end{pmatrix}} \mathbb{R}^1 \longrightarrow \mathbb{R}^0 \longrightarrow \mathbb{R}^0 \to \cdots$$

provides relations defining the subspace span $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \subset \mathbb{R}^3$, and furthermore these resolutions encode the meta-relations. We call such a sequence a *resolution*.

De Rham cohomology arises naturally from resolving the space of locally constant functions.

Why would we want to resolve the space of locally constant functions? It's useful to be able to analyze functions locally. For example, with smooth (=infinitely differentiable=C^∞) functions, we have the notion of a partition of unity:

Definition. Given an open cover $\{U_\alpha\}$ of a smooth manifold, there exist smooth bump functions $\{\phi_\alpha\}$ such that

- at every point, there is a neighborhood on which only finitely many ϕ_α are nonzero,
- $\sum_\alpha \phi_\alpha = 1$, and
- each $\phi_\alpha \geq 0$ with $\phi_\alpha > 0$ only inside U_α [specifically, the support of a function f is defined to be the closure $\text{supp}(f) := \overline{\{x | f(x) \neq 0\}}$, and $\text{supp}\phi_\alpha \subset U_\alpha$].

Such a collection of functions is called a *partition of unity*.

To analyze a function f, it's often easier to study $1 \cdot f = \sum_\alpha \phi_\alpha f$, so that each term $\phi_\alpha f$ is localized within U_α. The problem with a locally constant function f is that $\phi_\alpha f$ is no longer locally constant. Thus we are led to resolve locally constant functions in terms of smooth functions.

We begin on $\mathbb{R}^n$ with the tensor algebra, spanned by terms of the form

$$f \underbrace{dx^{i_1} \otimes \cdots \otimes dx^{i_k}}_{k \text{ factors}} \in T^k(\mathbb{R}^n), \quad \text{with } f \in C^\infty(\mathbb{R}^n).$$

Denote $T^\bullet(\mathbb{R}^n) := \oplus_{k \in \mathbb{Z}_{\geq 0}} T^k(\mathbb{R}^n)$. Note that $T^0 = C^\infty(\mathbb{R}^n)$. The total derivative $\nabla : T^\bullet(\mathbb{R}^n) \to T^{\bullet+1}(\mathbb{R}^n)$ is the operator

$$\nabla = \sum_{i=1}^n \frac{\partial}{\partial x^i} \otimes .$$

For example,

$$\nabla\left(x^1\, dx^2 \otimes dx^3\right) = dx^1 \otimes dx^2 \otimes dx^3.$$

Locally (globally) constant functions are characterized by solutions of $\nabla f = 0$ for $f \in T^0(\mathbb{R}^n)$. Our first guess for a resolution looks like

$$T^0(\mathbb{R}^n) \xrightarrow{\nabla} T^1(\mathbb{R}^n) \xrightarrow{\nabla} T^2(\mathbb{R}^n) \xrightarrow{\nabla} \cdots$$

but this does not satisfy the necessary condition $\nabla \circ \nabla = 0$. By symmetry of partial derivatives, and polarization,

$$\begin{aligned}
\nabla \circ \nabla &= \sum_{i,j} \frac{\partial^2}{\partial x^i \partial x^j}\, dx^i \otimes dx^j \otimes \\
&= \tfrac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x^i \partial x^j} \left((dx^i + dx^j) \otimes (dx^i + dx^j) - dx^i \otimes dx^i - dx^j \otimes dx^j \right) \otimes .
\end{aligned}$$

Thus we automatically satisfy the desired condition if we set $\alpha \otimes \alpha = 0$ for all $\alpha \in T^1(\mathbb{R}^n)$. Thus we define

$$\Omega^k(\mathbb{R}^n) := T^k(\mathbb{R}^n) / \{\alpha \otimes \alpha = 0\},$$

and with it we use the new notation

$$\otimes \mapsto \wedge, \quad \nabla \mapsto d := \sum_{i=1}^n \frac{\partial}{\partial x^i} \wedge,$$

so that

$$d^2 = 0, \quad \alpha \wedge \alpha = 0, \ \forall \alpha \in \Omega^1(\mathbb{R}^n).$$

This is antisymmetric since

$$0 = (\alpha_1 + \alpha_2) \wedge (\alpha_1 + \alpha_2) = \alpha_1 \wedge \alpha_2 + \alpha_2 \wedge \alpha_1.$$

Remark. Under general change of coordinates, the expression for d remains unchanged. However, changing coordinates for ∇, by the gauge principle, requires a connection. Normally one chooses the Levi-Civita connection associated with some metric.

Now we have arrived at the guess of a resolution

$$\Omega^0(\mathbb{R}^n) \xrightarrow{d} \Omega^1(\mathbb{R}^n) \xrightarrow{d} \Omega^2(\mathbb{R}^n) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(\mathbb{R}^n) \xrightarrow{d} 0 = \Omega^{n+1}(\mathbb{R}^n).$$

The *Poincaré lemma* is a computation which shows that this sequence is exact. Assuming this, we have achieved our goal of resolving locally constant functions. In summary, the only solutions over $\mathbb{R}^n$ to $d\omega = 0$ are the obvious solutions of the form $\omega = d\eta$, with the exception of ω equal to some constant function.

Manifolds

A *topological manifold of dimension n* is a set equipped with an n-dimensional atlas, which is Hausdorff and second-countable.

An *n–dimensional atlas* on a set X is a cover $\{U_\alpha\}$ of X, and charts $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ such that

- each $V_\alpha \subset \mathbb{R}^n$ is open,
- each ϕ_α is a bijection, and
- each transition function $\phi_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1} : V_\alpha \to V_\beta$ is a homeomorphism.

Remark. Abstractly, manifolds begin life as a set, and inherit all their properties from their atlas. For example, subset of a manifold is *open* if it is open in each chart.

Remark. Given two different atlases on the same set, if their union is still an atlas, then the atlases are called *compatible*, and the resulting manifolds are considered equivalent.

Definition. An manifold is *smooth* if the transition functions are required instead to be diffeomorphisms.

Remark. Functions on smooth manifolds are *smooth* if they are smooth in each chart.

Definition. A smooth manifold is *oriented* if all transition functions $\phi_{\alpha\beta}$ are *orientation-preserving*, i.e. they satisfy

$$\det\left(\frac{\partial}{\partial x^j}\phi^i_{\alpha\beta}\right) > 0.$$

Remark. It's complicated, but one can extend this definition to topological manifolds.

Definition. Given an oriented manifold X, we define the *orientation-reversed manifold $\overline{X}$* to be the same smooth manifold, but with each coordinate chart reflected.

Examples of exotic manifolds

Topological/smooth manifolds, together with continuous/smooth maps, form a *category*. This means that every manifold has an identity map, and maps can be composed. In any category, there is a notion of *isomorphism*, which is a map with a two-sided inverse.

Definition. A continuous map of topological manifolds $f : X_1 \to X_2$ is a *homeomorphism* if it is an isomorphism of topological manifolds, i.e. there exists a continuous $f^{-1} : X_2 \to X_1$ such that $f^{-1} \circ f = \mathrm{Id}_{X_1}$ and $f \circ f^{-1} = \mathrm{Id}_{X_2}$.

Definition. A smooth map of smooth manifolds $f : X_1 \to X_2$ is a *diffeomorphism* if it is an isomorphism of smooth manifolds.

It's easy to place multiple smooth structures on the same topological manifold. For example, consider two smooth atlases on the same copy of $\mathbb{R}$, giving two smooth manifolds which we denote by X_1 and X_2. On X_1 we use the atlas with the single chart $\phi = \mathrm{Id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$. On X_2 we use the single chart $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(x) = x^3$. Individually, these are each clearly smooth atlases, since the only transition function is the identity. These two atlases are compatible topologically, since $\psi \circ \phi^{-1} = x \mapsto x^3$ and $\phi \circ \psi^{-1} = x \mapsto x^{1/3}$ are homeomorphisms. Thus X_1 and X_2 are the same topological manifold. However, they are not smoothly the compatible, since $x^{1/3}$ is not smooth.

We should not get too excited, since X_1 and X_2 are diffeomorphic. In particular, the map $X_1 \to X_2$ given by $x \mapsto x^{1/3}$ is a diffeomorphism. (Remember, smoothness of a map is defined in terms of coordinate charts!)

What we really want to understand is the difference between *diffeomorphism classes* of smooth manifolds, and *homeomorphism classes* of topological manifolds. Visualizing examples is not easy, due to the following result:

Theorem (Moise's Theorem (with others))**.** *Let X be a topological manifold of dimension $d \leq 3$. Then X admits a smooth structure, unique up to diffeomorphism.*

The first examples of exotic smooth structures were discovered by Milnor on the 7-sphere S^7. There are 28 distinct smooth structures on S^7. They can be realized explicitly as the manifolds obtained by the equations

$$\begin{aligned} a^2 + b^2 + c^2 + d^3 + e^{6k-1} &= 0, \\ |a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 &= \varepsilon, \\ (a, b, c, d, e) &\in \mathbb{C}^5, \end{aligned}$$

for $\varepsilon > 0$ small, and $k = 1, \ldots, 28$. Perhaps it is best to view exotic structures as distinct manifolds which are "accidentally" homeomorphic.