Elliptic operators

Let X be a closed oriented manifold of dimension n. Let $E \to X$ and $F \to X$ be two Euclidean vector bundles. A *differential operator D of degree d* is an operator between smooth sections

$$D : \Gamma(X; E) \to \Gamma(X; F)$$

such that in local coordinates, D is of the form

$$D = \sum_{|\alpha| \leq d} a_\alpha(x) \partial_\alpha,$$

where a_α are smooth functions with values in $\mathrm{Hom}(E, F)$. Specifically, given local frames $\{e_i\}_{i=1}^k$ and $\{f_j\}_{j=1}^\ell$ of E and F respectively, we can locally write $s \in \Gamma(X; E)$ as $s = \sum_{i=1}^k s^i(x) e_i$ for smooth scalar functions $s^i(x)$. Then

$$Ds = \sum_{|\alpha| \leq d} \sum_{i,j} a_i^{\alpha j}(x) (\partial_\alpha s^i) f_j.$$

The replacement for the Fourier transform associated to D is called the *principal symbol* $\sigma(D)$ of D. It is a smooth function on T^*X which on each fiber is a homogeneous polynomial of degree d with coefficients in $\mathrm{Hom}(E, F)$. For local coordinates $x_1, \ldots, x_n$ on X, there are corresponding local coordinates on T^*X given by $(x_1, \ldots, x_n, p_1, \ldots, p_n)$, where p_i are the coordinates dual to the dx^i. The local formula for $\sigma(D)$ is

$$\sigma(D, x_1, \ldots, x_n, p_1, \ldots, p_n) = \sum_{|\alpha| = d} a^\alpha(x) p_\alpha.$$

This function is well-defined on T^*X and is independent of coordinates.

Example. Let $\Delta : \Omega^k(X) \to \Omega^k(X)$ be the Hodge Laplacian. Then

$$\sigma(\Delta, x, p) = -|p|^2 \, \mathrm{Id}_{\Lambda^k}.$$

Definition. A linear differential operator D is *elliptic* if $\sigma(D, x, p) \in \mathrm{Iso}(E, F)$ whenever $p \neq 0$.

Clearly the Hodge Laplacian is elliptic since $\sigma(\Delta, x, p)$ is a nonzero multiple of the identity when $p \neq 0$.

Consider the operator $d : \Omega^k(X) \to \Omega^{k+1}(X)$. The symbol is

$$\sigma(d, x, p) = p \wedge \bullet : \Lambda^k T_x^* X \to \Lambda^{k+1} T_x^* X,$$

thus d is *not* elliptic, since p is in the kernel of $p \wedge \bullet$.

Another example of an operator which is not elliptic is $d^* : \Omega^k(X) \to \Omega^{k-1}(X)$. Its symbol is

$$\sigma(d, x, p) = -i_p : \Lambda^k T_x^* X \to \Lambda^{k-1} T_x^* X,$$

where i_p is the contraction map is defined as the alternating sum

$$i_p(p_1 \wedge \cdots \wedge p_k) := \langle p, p_1 \rangle \, p_2 \wedge \cdots \wedge p_k - \langle p, p_2 \rangle \, p_1 \wedge p_3 \wedge \cdots \wedge p_k + \cdots.$$

In contrast, the operator $d + d^* : \Omega^{\text{even}}(X) \to \Omega^{\text{odd}}(X)$ is elliptic, since its symbol is

$$\sigma(d + d^*, x, p) = c(p) := -i_p(\bullet) + p \wedge \bullet : \Lambda^{\text{even}} T_x^* X \to \Lambda^{\text{odd}} T_x^* X.$$

Since $(d+d^*)^2 = \Delta$ and symbols compose as expected under composition of operators, $c(p)$ satisfies the Clifford algebra relation

$$c(p)^2 = -|p|^2 \operatorname{Id}_{\Lambda^\bullet},$$

and is thus invertible (with inverse $-c(p/|p|^2)$) whenever $p \neq 0$.

If D is an elliptic operator, then its formal adjoint D^* is given by

$$D^* = \sum_{|\alpha| \le d} (-1)^{|\alpha|} \partial_\alpha a_\alpha^t(x),$$

where a_α^t is the transpose of a_α. The multiplication operator a_α^t and differentiation operator ∂_α commute modulo operators of order $d - 1$. Thus $\sigma(D^*) = (-1)^d \sigma(D)^t$. Furthermore, D is elliptic iff D^* is elliptic.

Elliptic operators have finite-dimensional kernels, and like we saw in the Hodge decomposition, their images are closed. Furthermore, the cokernel $\Gamma(F)/(\text{image } D)$ can be identified with $(\text{image } D)^\perp = \ker D^*$ which is also finite-dimensional. Thus we have a well defined *index*

$$\operatorname{ind}(D) = \dim \ker(D) - \dim \operatorname{coker}(D).$$

The Atiyah-Singer index theorem gives an explicit cohomological formula to compute $\operatorname{ind}(D)$ for any elliptic operator D in terms of cohomology. For example,

$$\begin{aligned} \operatorname{ind}(d + d^* : \Omega^{\text{even}}(X) \to \Omega^{\text{odd}}(X)) &= \dim \ker(d + d^*) - \dim \ker((d + d^*)^*) \\ &= \dim \mathcal{H}^{\text{even}}(X) - \dim \mathcal{H}^{\text{odd}}(X) \\ &= \chi(X) \end{aligned}$$

is cohomological.

The theory of pseudodifferential operators, which is beyond the scope of this course, allows one to show that if D is an elliptic differential operator of order D, then there exists a pseudodifferential operator Q of order $-d$ (roughly, constructed by inverting the symbol of D) which continuously maps the Sobolev spaces $H^s \to H^{s+d}$, and such that

$$QD = \operatorname{Id} + K,$$

where K is a smoothing operator. Specifically, we mean that $K : H^s \to C^\infty$, and for each s' we have an estimate of the form $\|Kf\|_{s'} \le C_{K,s,s'} \|f\|_s$. We now deduce the elliptic estimate

$$\|f\|_{H^{s+d}} \le C_{D,s} \left(\|Df\|_{H^s} + \|f\|_{H^s} \right)$$

as follows:

$$\|f\|_{H^{s+d}} = \|QDf - Kf\|_{H^{s+d}} \le C_{Q,s} \|Df\|_{H^s} + C_{K,s,s+d} \|f\|_{H^s}.$$

Proof of outstanding lemmas

Recall our definition of Sobolev spaces over the n-torus T^n based on the Fourier coefficients, given by

$$\begin{aligned} H^s(T^n) &:= \left\{\{c_k\} \mid \sum (1+|k|)^{2s} |c_k|^2 < \infty \right\} \\ &= \left\{\{c_k\} \mid (1+|k|)^s c_k \in \ell^2(\mathbb{Z}^n)\right\} \\ &= (1+|k|)^{-s}\, \ell^2(\mathbb{Z}^n). \end{aligned}$$

It's easy to verify that

$$\partial_i : H^s \to H^{s-1}$$

is continuous, $(1+\Delta) : H^s \to H^{s-2}$ is an isomorphism, and more generally, $(\Delta - \lambda) : H^s \to H^{s-2}$ for all λ which are *not* eigenvalues of Δ. Finally, the Riesz representation theorem states that the L^2 pairing

$$\langle f, g\rangle_{L^2} = \sum_k c_k \overline{d}_k$$

is a perfect pairing, meaning that every element of $(L^2)^*$ arises as the inner product with some element of L^2. If $f \in H^s(T^n)$ with Fourier series $\{c_k\}$, and if $g \in H^{-s}(T^n)$ with Fourier series $\{d_k\}$, then $\langle f, g\rangle_{L^2}$ makes sense because

$$"\langle f, g\rangle_{L^2}" = \sum_k c_k \overline{d}_k = \sum_k (1+|k|)^s c_k \; (1+|k|)^{-s} \overline{d}_k \le \|f\|_{H^s} \|g\|_{H^{-s}} < \infty,$$

where the $\le$ is Cauchy-Schwarz. Moreover, since $\{(1+|k|)^s c_k\}$ is in $\ell^2(\mathbb{Z}^n)$, any element of $(H^s)^*$ is represented as an inner product with some $\{(1+|k|)^{-s} d_k\}$ in $\ell^2(\mathbb{Z}^n)$, i.e.

$$(H^s)^* = H^{-s}.$$

Definition. A *distribution* on T^n is an element of

$$\mathcal{D}(T^n) := \bigcup_{s \to -\infty} H^s(T^n).$$

Remark. For any $f \in \mathcal{D}(T^n)$, there is some (potentially very negative) s such that $f \in H^s(T^n)$. Consequently, via the L^2 pairing, f determines a continuous linear functional on $H^{-s}(T^n)$, which restricts to a continuous linear functional on $C^\infty(T^n)$ (continuous with respect to each C^k-norm, or equivalently each H^s-norm). The more common definition of a distribution is an element of the continuous dual $\mathcal{D}(X) := (C^\infty(X))^*$, but this description over T^n in terms of Fourier series is much more transparent. For general manifolds X, we can obtain a Fourier description of $\mathcal{D}(X)$ by using a partition of unity, and coordinate charts which map to small open subsets of T^n instead of $\mathbb{R}^n$.

Remark. If $f \in \mathcal{D}(T^n)$, then $f \in H^s(T^n)$ for some s, and thus $G^k f \in H^{s+2k}(T^n)$ is continuous for sufficiently large k. It follows that (up to a constant function) $f = \Delta^k g$ for some $g \in C^0(T^n)$. Thus any distribution can be described in terms of the distributional derivatives of some continuous function.

Now let's finish the remaining analytical details. For the elliptic estimate on T^n,

$$\|f\|_{H^s} + \|\Delta f\|_{H^s} = \|c_k\|_{H^s} + \left\||k|^2 c_k\right\|_{H^s} \sim \|(1+|k|)^s c_k\|_{L^2} + \left\||k|^2 (1+|k|)^{s+2} c_k\right\|_{L^2} \sim \|f\|_{H^{s+2}},$$

so

$$\|f\|_{H^{s+2}} \leq C\left(\|f\|_{H^s} + \|\Delta f\|_{H^s}\right).$$

For the Sobolev embedding $H^s(T^n) \hookrightarrow C^0(T^n)$ for $s > n/2$, note that $\|f\|_{H^s} = \sum(1+|k|)^{2s}|c_k|^2 < \infty$, so

$$\begin{aligned}
\|f\|_{C^0} &= \left\|\sum_k c_k e^{ik\cdot x}\right\|_{C^0} \\
&\leq \sum_k |c_k| \\
&= \sum_k (1+|k|)^s |c_k| (1+|k|)^{-s} \\
&\leq \sqrt{\sum_k (1+|k|)^{2s}|c_k|^2}\sqrt{\sum_k (1+|k|)^{-2s}} \\
&= C\|f\|_{H^s}, \quad C = \|(1+|k|)^{-s}\|_{\ell^2(\mathbb{Z}^n)},
\end{aligned}$$

where the second inequality is Cauchy-Schwarz. The constant C is finite because $(1+|k|)^{-s} \in \ell^2$ iff $s > n/2$.

Note that strict inequality is necessary, because one can check that on the n-ball of radius $r = \frac{1}{2}$, the radial function $\ln\ln(1/r)$ is in $H^{n/2}$. But clearly it is not in C^0 since it is unbounded as $r \to 0$.

For the compactness of $H^s(T^n) \hookrightarrow H^{s'}(T^n)$ whenever $s > s'$, we need to show that given a sequence $\{f_i\}$ with Fourier coefficients $\{c_{i,k}\}$, and if $\sum_k (1+|k|)^{2s}|c_{i,k}|^2 \leq C$, then $\left\{(1+|k|)^{s'}c_{i,k}\right\}$ has a subsequence which converges in $\ell^2(\mathbb{Z}^n)$. By absorbing $(1+|k|)^s$ into $c_{i,k}$, we reduce to the case $s = 0$ and $s' < 0$. The point is that if we restrict to finitely many k at one time, we can pass to a convergent subsequence since the unit ball in a finite-dimensional Hilbert space is compact. Since the weights $(1+|k|)^{s'} \to 0$ as $|k| \to \infty$, we are able to pass to a subsequence for which $\left\{(1+|k|)^{s'}c_{i,k}\right\}$ is Cauchy in $\ell^2(\mathbb{Z}^n)$. Specifically, for any integer I, we can demand a subsequence such that for all $i_1, i_2 \geq I$,

$$\sum_{|k|\leq I} |c_{i_1,k} - c_{i_2,k}|^2 \leq 1/I.$$

Furthermore, we can pass to a subsequence where this is true for each I. Then we need to control

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} (1+|k|)^{2s'}|c_{i_1,k} - c_{i_2,k}|^2 \\
&= \sum_{|k|\leq I} (1+|k|)^{2s'}|c_{i_1,k} - c_{i_2,k}|^2 + \sum_{|k|>I} (1+|k|)^{2s'}|c_{i_1,k} - c_{i_2,k}|^2 \\
&\leq 1/I + (1+I)^{2s'} \cdot C \to 0 \text{ as } I \to \infty,
\end{aligned}$$

hence our chosen subsequence is Cauchy.