Recall from last time, we derived the “exact sequence”

$$\overbrace{\check{C}^0(X;P)}^{\substack{\text{local}\\ \text{trivializations}\\ \{\phi_\alpha\}}}$$

$$\check{C}^0(X;G_{C^\infty}) \overset{\{\phi_\alpha\}\cdot\{g_\alpha\}:=\{\phi_\alpha g_\alpha\}}{\rightsquigarrow} \check{C}^0(X;P) \qquad \check{C}^0(X;P) \xrightarrow{d=\{\phi_\alpha\}\mapsto\{\phi_{\alpha\beta}:=\phi_\alpha^{-1}\phi_\beta\}} \check{Z}^1(X;G_{C^\infty})$$

$$0 \longrightarrow \underbrace{\check{Z}^0(X;\mathrm{Aut}(P))}_{\substack{\text{bundle}\\ \text{automorphisms}\\ \{g_\alpha\}|g_\alpha=\phi_{\alpha\beta}g_\beta\phi_{\alpha\beta}^{-1}}} \longrightarrow \underbrace{\check{C}^0(X;G_{C^\infty})}_{\substack{\text{changes of}\\ \text{trivialization}\\ \{g_\alpha\}}} \overset{\{\phi_{\alpha\beta}\}\cdot\{g_\alpha\}:=\{g_\alpha^{-1}\phi_{\alpha\beta}g_\beta\}}{\rightsquigarrow} \underbrace{\check{Z}^1(X;G_{C^\infty})}_{\substack{\text{transition}\\ \text{functions}\\ \{\phi_{\alpha\beta}\}|\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta}=1}} \longrightarrow \underbrace{\check{H}^1(X;G_{C^\infty})}_{\substack{\text{isomorphism}\\ \text{classes of smooth}\\ \text{principal } G\text{-bundles}}} \longrightarrow 0$$

Note that bundle automorphisms transform via the adjoint representation. Also, local trivializations correspond to local sections of P. In particular, a global section of P corresponds to a global trivialization of P, so P has global sections iff it is isomorphic to the trivial principal bundle $P \cong X \times G$.

Another important question is when does Čech cohomology with smooth coefficients $\check{H}^k(X;G_{C^\infty})$ coincide with ordinary cohomology $H^k(X;G)$ with locally constant coefficients. In order for $H^k(X;G)$ to make sense, G should be abelian. In this case, in order to naturally identify $\check{H}^k(X;G_{C^\infty})$ with $H^k(X;G)$, we want to identify the sheaf G_{C^∞} with the locally constant sheaf G. But G_{C^∞} corresponds with the locally constant sheaf whenever G has the discrete topology. For example, smooth $\mathbb{Z}$-valued functions are necessarily locally constant. In contrast, $\check{H}^k(X;\mathbb{R}_{C^\infty}) = 0$ for $k > 0$ thanks to partitions of unity, while $H^k(X;\mathbb{R})$ is usually nontrivial.

Reduction of structure group

Suppose P is a principal G-bundle, and $H \subset G$ is a subgroup (not necessarily normal). Then we have a “short exact sequence”

$$0 \to H \to G \to G/H \to 0.$$

(When H is not normal, we should think of “exactness” as the property that each coset of G has a free and transitive action of H.)

To keep in mind a concrete example, consider

$$0 \to \mathrm{O}(k) \to \mathrm{GL}(k) \to \mathrm{Met}(k) \to 0,$$

where

$$\mathrm{Met}(k) := \{\text{symmetric positive-definite matrices}\} = \mathrm{GL}(k)/\mathrm{O}(k).$$

How is $\mathrm{GL}(k)/\mathrm{O}(k)$ identified with positive-definite matrices? Consider the map $\mathrm{GL}(k) \to \mathrm{Met}(k)$ is given by $g \mapsto (g^T)^{-1}\,\mathrm{Id}_{k\times k}\,g^{-1} = \rho_{\mathrm{Met}}(g)\,\mathrm{Id}_{k\times k}$. (What’s the reason for this transformation law?

Hint: we want the inner product of two vectors to be independent of frame.) Clearly if h is orthogonal, then gh has the same image as g, so we get a well-defined map from left cosets $\mathrm{GL}(k)/\mathrm{O}(k) \to \mathrm{Met}(k)$.

Exercise. Verify that the map $M \mapsto M^{-1/2}$ is the two-sided inverse $\mathrm{Met}(k) \to \mathrm{GL}(k)/\mathrm{O}(k)$ by assuming the polar decomposition $g = ph$ for any $g \in \mathrm{GL}(k)$, p positive-definite, and h orthogonal.

The associated bundle $P \times_{\rho_L} \mathrm{GL}(k)/\mathrm{O}(k) = P \times_{\rho_{\mathrm{Met}}} \mathrm{Met}(k)$ then corresponds to the bundle of Euclidean metrics on the corresponding fibers. A global section

$$s \in \check{H}^0(X; P \times_{\rho_{\mathrm{Met}}} \mathrm{Met}(k)) = \Gamma(X; P \times_{\rho_{\mathrm{Met}}} \mathrm{Met}(k))$$

corresponds to a metric on the associated standard vector bundle $E = P \times_{\rho_{\mathrm{st}}} \mathbb{R}^k$. A metric then determines an $\mathrm{O}(k)$ structure on P. It picks out the subset of local trivializations which are orthonormal, and upon restriction to these, the transition functions take values in $\mathrm{O}(k)$.

More abstractly, given the principal G-bundle P, we seek to modify it so that the transition functions take values in H. Specifically, suppose $\{\phi_\alpha\}$ are local trivializations such that $\{\phi_{\alpha\beta}\} \in \check{Z}^1(X; G_{C^\infty})$. We seek a change of trivialization $\{g_\alpha\} \in \check{C}^0(X; G_{C^\infty})$ such that $\{\phi_{\alpha\beta}\} \cdot \{g_\alpha\}$ belongs to $\check{Z}^1(X; H_{C^\infty})$. Thus $g_\alpha^{-1}\phi_{\alpha\beta}g_\beta = h_{\alpha\beta}$ for some $h_{\alpha\beta}$ with values in H. Equivalently, $g_\alpha = \phi_{\alpha\beta}g_\beta h_{\alpha\beta}^{-1} \in \phi_{\alpha\beta}g_\beta H$, so

$$\{g_\alpha H\} \in \check{H}^0(X; P \times_{\rho_L} G/H).$$

Conversely, given any section of $P \times_{\rho_L} G/H$, it's easy to see that if we can locally locally we can lift to $\{g_\alpha\} \in \check{C}^0(X; G_{C^\infty})$, then the corresponding

$$h_{\alpha\beta} := g_\alpha^{-1}\phi_{\alpha\beta}g_\beta \in C^\infty(U_{\alpha\beta}; H)$$

so that

$$\{h_{\alpha\beta}\} \in \check{Z}^1(X; H_{C^\infty})$$

determines a principal H-bundle. Homologically, we have the following "exact sequence":

$$\mathcal{G}_P = \underbrace{\check{H}^0(X; P \times_{\mathrm{Ad}} G)}_{\{g_\alpha\} | g_\alpha = \phi_{\alpha\beta}g_\beta\phi_{\alpha\beta}^{-1}} \xrightarrow{\{\tilde{g}_\alpha\}\cdot\{g_\alpha H\} := \{\tilde{g}_\alpha g_\alpha H\}} \underbrace{\check{H}^0(X; P \times_{\rho_L} G/H)}_{\substack{\text{reductions} \\ \{g_\alpha H\} | g_\alpha H = \phi_{\alpha\beta}g_\beta H}} \xrightarrow{\{g_\alpha H\} \mapsto \{g_\alpha^{-1}\phi_{\alpha\beta}g_\beta\}} \check{H}^1(X; H_{C^\infty})$$

$$\check{H}^1(X; H_{C^\infty}) \to \check{H}^1(X; G_{C^\infty}) \longrightarrow \check{H}^1(X; (G/H)_{C^\infty})$$

Note that $\check{H}^1(X; (G/H)_{C^\infty})$ only really makes sense when H is normal, since otherwise there is no clear interpretation of the cocycle condition $\phi_{\beta\gamma}H\left(\phi_{\alpha\gamma}H\right)^{-1}\phi_{\alpha\beta}H = 1$. In this case, $P \times_{\rho_L} G/H$ is the associated principal G/H bundle. Homological algebra dictates that reductions should only exist when the corresponding principal bundle $P \times_{\rho_L} G/H$ is trivial. Indeed, reductions correspond to global sections of this principal bundle, so they exist iff it is trivial.

Regardless of whether or not H is normal, what really counts is the space of reductions $\Gamma(X; P \times_{\rho_L} G/H)$. These reductions, up to the action by gauge transformations, parameterize the isomorphism classes of smooth principal H-bundles over P.

A more sophisticated application of this formalism is the following:

Theorem. *Over a complex manifold X, a reduction from a smooth vector bundle $E \to X$ to a holomorphic vector bundle is equivalent to a $\overline{\partial}_\alpha$-operator on E which satisfies $\overline{\partial}_\alpha^2 = 0$.*

To understand why, consider the sequence of sheaves given by

$$0 \to \mathcal{O}(\mathrm{GL}(k;\mathbb{C})) \to C^\infty(\mathrm{GL}(k;\mathbb{C})) \to \mathrm{Hol}(k) \to 0,$$

where $\mathcal{O}(\mathrm{GL}(k;\mathbb{C}))$ denotes the sheaf of holomorphic functions valued in $\mathrm{GL}(k;\mathbb{C})$, and $\mathrm{Hol}(k)$ denotes the space of operators

$$\overline{\partial}_\alpha : \Omega^0(U;\mathbb{C}^k) \to \Omega^{0,1}(U;\mathbb{C}^k)$$

subject to the additional constraints

- $\overline{\partial}_\alpha(f\,s) = \left(\overline{\partial} f\right) s + f\,\overline{\partial}_\alpha s$ for all $f \in \Omega^0(U;\mathbb{C})$ and $s \in \Omega^0(U;\mathbb{C}^k)$,
- $0 = \overline{\partial}_\alpha^2 s \in \Omega^{0,2}(U;\mathbb{C})$ for all $s \in \Omega^0(U;\mathbb{C}^k)$.

The map $C^\infty(\mathrm{GL}(k)) \to \mathrm{Hol}(k)$ is given by

$$g \mapsto g \circ \overline{\partial} \circ g^{-1} = g \circ \left(\overline{\partial} g^{-1} + g^{-1}\overline{\partial}\right) = \overline{\partial} - (\overline{\partial} g)g^{-1},$$

where $\overline{\partial}$ is the coordinatewise standard $\overline{\partial}$ operator.

Assuming the exactness of the above short exact sequence of sheaves, we expect an exact sequence in Čech cohomology of the form

$$\check{H}^0(X; P \times_\rho \mathrm{Hol}(k)) \to \check{H}^1(X;\mathrm{GL}(k;\mathbb{C})_{\mathcal{O}}) \to \check{H}^1(X;\mathrm{GL}(k;\mathbb{C})_{C^\infty}).$$

Here we have principal bundles $\check{H}^1(X;\mathrm{GL}(k)_{\mathcal{O}})$ which correspond to holomorphic vector bundles, i.e. vector bundles with holomorphic transition functions. Then $\check{H}^0(X; P \times_\rho \mathrm{Hol}(k))$ corresponds to a $\overline{\partial}_\alpha$ operator on a smooth vector bundle P, whose kernel selects the "holomorphic sections."

The bulk of the work for this picture amounts to showing exactness of:

$$0 \to \mathcal{O}(\mathrm{GL}(k;\mathbb{C})) \to C^\infty(\mathrm{GL}(k;\mathbb{C})) \to \mathrm{Hol}(k) \to 0.$$

Exactness at the left is obvious, since holomorphic functions are a subspace of smooth functions. Exactness at the center is simply the statement that

$$g\overline{\partial} g^{-1} = \tilde{g}\overline{\partial}\tilde{g}^{-1} \iff \overline{\partial}(g^{-1}\tilde{g}) = 0.$$

The hardest part is surjectivity. The condition $\overline{\partial}_\alpha^2 s = 0$ is an integrability condition which ensures that we can find g such that $\overline{\partial}_\alpha = g\overline{\partial} g^{-1}$. For details of the integrability theorem, see Donaldson and Kronheimer, 2.1.53.

Classification of principal bundles on a 4-manifold

A connected Lie group G is called *simple* if it is non-abelian, and the Lie algebra $\mathfrak{g}$ of G has no non-trivial ideals besides $0, \mathfrak{g}$. For example, $\mathrm{U}(1)$ is not simple since it is abelian. More generally, $\mathrm{U}(k)$ is not simple since its Lie algebra contains $\mathfrak{u}(1)$ as an ideal. However, $\mathrm{SU}(k)$ is simple. The special orthogonal groups $\mathrm{SO}(k)$ are simple for $k = 3$ and $k \geq 5$. Compact simple simply-connected Lie groups $\tilde{G}$ are in bijection with admissible Dynkin diagrams. Classification of Dynkin diagrams yields the list of possible groups

$$\tilde{G} \in \{\mathrm{SU}(k), \mathrm{Spin}(k), \mathrm{Sp}(2k), E_6, E_7, E_8, F_4, G_2\}.$$

The center $Z(\tilde{G})$ is the subgroup of elements which commute with everything else, and is determined by

$$Z(\tilde{G}) = \Lambda_{\text{weight}}/\Lambda_{\text{root}},$$

which is a finite abelian group.

More generally, any compact simple non-simply-connected Lie group G is the quotient of $\tilde{G}$ by a subgroup of its center. This subgroup lifts to an intermediate lattice given by the kernel of the exponential map:

$$\Lambda_{\text{root}} \subset \ker(\exp) \subset \Lambda_{\text{weight}}.$$

Then $Z(G) = \Lambda_{\text{weight}}/\ker(\exp)$, and $\pi_1(G) = \ker(\exp)/\Lambda_{\text{root}}$ are both finite abelian groups.

For example, consider $\tilde{G} = \mathrm{SU}(k)$. A maximal torus $T \subset \tilde{G}$ is the diagonal matrices of determinant 1, which is a copy of $\mathrm{U}(1)^{k-1} \subset \mathrm{U}(1)^k$. The center $Z(\tilde{G})$ is isomorphic to $\mathbb{Z}_k$, consisting of multiples of the identity matrix $e^{2\pi i/k}I$. The Lie algebra $\mathfrak{t}$ of T is $\{(\theta_1, \ldots, \theta_{n+1}) \in \mathbb{R}^{n+1} | \sum \theta_i = 0\}$. The root lattice is the kernel of the exponential map, which is the subset intersecting $2\pi\mathbb{Z}^{n+1}$. The weight lattice is where the exponential map hits the center, i.e. the intersection of $\mathfrak{t}$ with

$$2\pi\mathbb{Z}^{n+1} + 2\pi/k\,(1, \ldots, 1)\,\mathbb{Z}.$$

The group $\mathrm{PU}(k) := \mathrm{SU}(k)/Z(\mathrm{SU}(k))$ has fundamental group $\pi_1(\mathrm{PU}(k)) = Z(\mathrm{SU}(k)) = \mathbb{Z}_k$. Note that $\mathrm{Spin}(3) = \mathrm{SU}(2)$, and $\mathrm{PU}(2) = \mathrm{SU}(2)/\mathbb{Z}_2 = \mathrm{SO}(3)$.