If $\phi \in \check{C}^p(X;\mathcal{S})$, this means that on some open cover $\{U_\alpha\}$, there is a collection of functions $\left\{\phi_{\alpha_0\cdots\alpha_p} \in \mathcal{S}(U_{\alpha_0\cdots\alpha_p})\right\}$, where the sheaf $\mathcal{S}$ determines some class of functions on each overlap $U_{\alpha_0\cdots\alpha_p} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$. Whenever $\mathcal{S}$ is a sheaf of abelian groups, then

$$d\phi = \left\{\phi_{\alpha_0\cdots\alpha_{p+1}} := \sum_{k=0}^{p+1}(-1)^k \phi_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_{p+1}}\right\}.$$

In other cases, we must find a suitable alternative interpretation in the same spirit.

Given a system of local trivializations $\{\phi_\alpha\}$ of a fiber bundle, each ϕ_α is equivalent to a local section of the principal bundle P over U_α. Thus a local trivialization is equivalent to an element of $\check{C}^0(X;P)$.

The transition functions are $\left\{\phi_{\alpha\beta} := \phi_\alpha^{-1}\phi_\beta\right\}$, which should be thought of as a Čech differential

$$\left\{\phi_{\alpha\beta}\right\} = d\left\{\phi_\alpha\right\} \in \check{C}^1(X;G_{C^\infty}),$$

with smooth G-valued functions (as opposed to locally-constant G-valued functions).

Not all elements of $\check{C}^1(X;G_{C^\infty})$ arise in this way from some principal bundle. The relation $d^2 = 0$ still holds if we interpret

$$d\left\{\phi_{\alpha\beta}\right\} := \left\{\phi_{\alpha\beta\gamma} := \phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta}\right\}.$$

The condition

$$\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta} = \mathrm{Id}$$

is the *cocycle condition* for transition functions. Usually the constraints are written with three formulas, but this form encompases them all. Setting $\alpha = \beta = \gamma$, we get

$$\mathrm{Id} = \phi_{\alpha\alpha}\phi_{\alpha\alpha}^{-1}\phi_{\alpha\alpha} = \phi_{\alpha\alpha}.$$

Setting only $\gamma = \alpha$, we get

$$\phi_{\beta\alpha}\phi_{\alpha\alpha}^{-1}\phi_{\alpha\beta} = \mathrm{Id} \implies \phi_{\beta\alpha} = \phi_{\alpha\beta}^{-1}.$$

Finally,

$$\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta} = \mathrm{Id} \implies \phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = \mathrm{Id}.$$

It's convenient to introduce the notation $\ker d := \check{Z}^p \subset \check{C}^p$ where $\check{Z}^p$ denotes the *cocycles* which are the cochains with trivial differential. In summary, we have a map

$$\begin{gathered} d : \check{C}^0(X;P) \to \check{Z}^1(X;C^\infty(G)). \\ \text{local trivializations} \mapsto \underset{\text{(satisfying cocycle condition)}}{\text{transition functions}} \end{gathered}$$

We now proceed to systematically develop bundle theory by asking natural homological questions.

Any element $\left\{\phi_{\alpha\beta}\right\} \in \check{Z}^1(X;C^\infty(G))$ is represented as the image of d from some principal bundle P'. In particular, the cocycle condition is precisely the consistency condition required to carry out the gluing construction $P' := \coprod_\alpha U_\alpha \times G/\sim$ which realizes the transition functions $\left\{\phi_{\alpha\beta}\right\}$.

Next we should ask how the choice of local trivialization affects the transition functions. As a guiding principle, we will discover an analogue of the exact sequence in ordinary cohomology:

$$0 \to Z^{i-1} \hookrightarrow C^{i-1} \xrightarrow{d} Z^i \twoheadrightarrow H^i \to 0.$$

We will interpret an analogue of this with $i = 1$ and coefficients in G_{C^∞}.

We call an element of $\check{C}^0(X; G_{C^\infty})$ a *change of trivialization*. To explain, we obtain a right action of $\check{C}^0(X; C^\infty(G))$ on $\check{C}^0(X; P)$ by

$$\{\phi_\alpha\} \cdot \{g_\alpha\} := \{\phi_\alpha g_\alpha\}.$$

Furthermore, the action is transitive (one orbit) since, after possibly refining the open cover, any $\{\phi'_\alpha\}$ is obtained by $\{\phi'_\alpha\} = \{\phi_\alpha\} \cdot \{(\phi_\alpha^{-1}\phi'_\alpha)\}$.

There is also a right action of $\check{C}^0(X; G_{C^\infty})$ on $\check{C}^1(X; G_{C^\infty})$ given by $\{\phi_{\alpha\beta}\} \cdot \{g_\alpha\} := \{g_\alpha^{-1}\phi_{\alpha\beta}g_\beta\}$. Furthermore, under these actions the Čech differential is equivariant:

$$d\left(\{\phi_\alpha\} \cdot \{g_\alpha\}\right) = d\left(\{\phi_\alpha g_\alpha\}\right) = \left\{g_\alpha^{-1}\phi_\alpha^{-1}\phi_\beta g_\beta\right\} = d\left(\{\phi_\alpha\}\right) \cdot \{g_\alpha\}.$$

In summary,

$$\begin{array}{ccc} \check{C}^0(X;P) & & \\ \wr\!\uparrow & \overset{d}{\searrow} & \\ \check{C}^0(X;G_{C^\infty}) & \rightsquigarrow & \check{Z}^1(X;G_{C^\infty}) \end{array}$$

where a squiggly arrow indicates a group action instead of an actual map. Note that since $\check{C}^0(X; P)$ consists of a single orbit, by equivariance, the image of d is the corresponding orbit in $\check{Z}^1(X; G_{C^\infty})$.

Suppose two principal bundles P_1 and P_2 share a common element in $\check{Z}^1(X; G_{C^\infty})$. Then P_1 and P_2 correspond to the same $\check{C}^0(X; G_{C^\infty})$ orbit. Furthermore, they are both isomorphic to the gluing construction P'. Thus P_1 and P_2 must be isomorphic. This establishes a correspondence

$$\begin{smallmatrix}\text{iso classes of}\\ \text{principal } G\text{-bundles}\end{smallmatrix} = \frac{\check{Z}^1(X; G_{C^\infty})}{\text{action of } \check{C}^0(X; G_{C^\infty})} =: \check{H}^1(X; G_{C^\infty}).$$

$$\begin{array}{ccccccc} \check{C}^0(X;P) & & & & & & \\ \wr\!\uparrow & \overset{d}{\searrow} & & & & & \\ \check{C}^0(X;G_{C^\infty}) & \rightsquigarrow & \check{Z}^1(X;G_{C^\infty}) & \longrightarrow & \check{H}^1(X;G_{C^\infty}) & \longrightarrow & 0 \end{array}$$

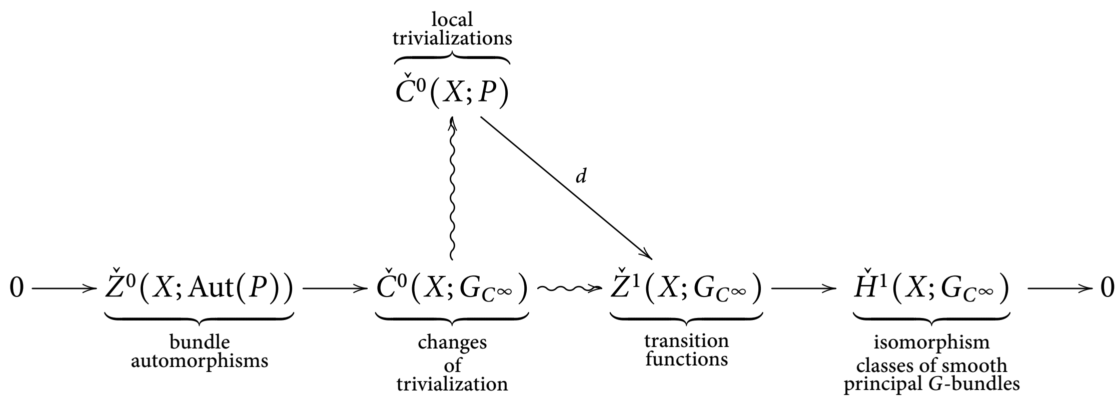
The next question is when does a change of trivialization act trivially on transition functions. For a meaningful answer to this question, we should fix local trivializations $\{\phi_\alpha\} \in \check{C}^0(X; P)$. Then it could happen that our change of trivialization $\{g_\alpha\} \in \check{C}^0(X; G_{C^\infty})$ satisfies

$$\begin{aligned} g_\alpha^{-1}\phi_{\alpha\beta}g_\beta &= \phi_{\alpha\beta} \\ \iff g_\alpha &= \phi_{\alpha\beta}g_\beta\phi_{\alpha\beta}^{-1}, \end{aligned}$$

i.e. the g_α transform via the adjoint representation. To cast this in the correct language, we should interpret $\check{Z}^0$ for any associated bundle

$$\{f_\alpha \in C^\infty(U_\alpha, F)\} \in \check{Z}^0(X; P \times_\rho F) = \check{H}^0(X; P \times_\rho F)$$

not in terms of the kernel of some differential, but rather as a collection of local sections which agree on the overlaps, thus defining a global section. In corresponding trivializations, $f_\alpha = \rho(\phi_{\alpha\beta}) f_\beta$. Thus in our case, $\{g_\alpha\} \in \check{Z}^0(X; P \times_{\mathrm{Ad}} G) = \check{Z}^0(X; \mathrm{Aut}(P)) = \mathcal{G}_P$. So the changes of trivialization which preserve transition functions are the gauge transformations.



We should mention the caveat that our identification of $\check{Z}^0(X;\mathrm{Aut}(P))$ with a subset of $\check{C}^0(X;G_{C^\infty})$ depends on the choice of local trivializations $\{\phi_\alpha\}$.

Sequences from coefficients

In ordinary cohomology, a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ gives a short exact sequence of chain complexes $0 \to C^\bullet(X;A) \to C^\bullet(X;B) \to C^\bullet(X;C) \to 0$ which gives a long exact sequence of cohomology

$$\cdots \to H^i(X;A) \to H^i(X;B) \to H^i(X;C) \to H^{i+1}(X;A) \to H^{i+1}(X;B) \to \cdots$$

We can imitate this with principal bundles. For example, consider

$$0 \to \mathbb{Z} \to \mathbb{R} \xrightarrow{e^{2\pi i x}} \mathrm{U}(1) \to 0.$$

This gives

$$\cdots \to \check{H}^1(X;\mathbb{R}_{C^\infty}) \to \check{H}^1(X;\mathrm{U}(1)_{C^\infty}) \to \check{H}^2(X;\mathbb{Z}) \to \check{H}^2(X;\mathbb{R}_{C^\infty}) \to \cdots$$

Beware that $\check{H}^p(X;\mathbb{R}_{C^\infty})$ is *not* equivalent to $H^p(X;\mathbb{R})$. Resolving the space of locally constant $\mathbb{R}$-valued functions is much more interesting than resolving the space of *smooth* $\mathbb{R}$-valued functions. Since $C^\infty(\mathbb{R})$ already has a partition of unity, it is resolved by

$$C^\infty(X;\mathbb{R}) \to 0 \to 0 \to \cdots.$$

Thus $\check{H}^0(X;\mathbb{R}_{C^\infty}) = C^\infty(X;\mathbb{R})$, and $\check{H}^p(X;\mathbb{R}_{C^\infty}) = 0$ for other p. On the other hand, since $\mathbb{Z}$ has the discrete topology, smooth $\mathbb{Z}$-valued functions are locally constant. Thus

$$0 \to \check{H}^1(X;\mathrm{U}(1)_{C^\infty}) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \to 0$$

so the group of isomorphism classes of smooth principal $\mathrm{U}(1)$ bundles is equivalent to the group $H^2(X;\mathbb{Z})$. Via the standard representation, principal $\mathrm{U}(1)$ bundles correspond to complex line bundles. Group composition on $\check{H}^1(X;\mathrm{U}(1)_{C^\infty})$ corresponds to multiplying together the $\mathrm{U}(1)$-valued transition functions, which is equivalent to tensor product. The isomorphism labeled c_1 is called the first Chern class. The homological algebra makes c_1 explicit. The recipe to compute it is as follows.

- Fix a "good cover" $\{U_\alpha\}$ of X so that all intersections are contractible.
- Given an isomorphism class $[P]$, choose a representative P. Pick local trivializations to obtain transition functions $\{\phi_{\alpha\beta}\} \in \check{Z}^1(X;\mathrm{U}(1)_{C^\infty})$ for P.
- Since each U_α is contractible, we can choose branches for $\{\eta_{\alpha\beta} = (2\pi i)^{-1}\log\phi_{\alpha\beta}\} \in \check{C}^1(X;\mathbb{R}_{C^\infty})$.
- Consider $d\{\eta_{\alpha\beta}\} = \{\eta_{\alpha\beta\gamma} = \eta_{\beta\gamma} - \eta_{\alpha\gamma} + \eta_{\alpha\beta}\}$. By the cocycle condition for $\{\phi_{\alpha\beta}\}$, $e^{2\pi i\eta_{\alpha\beta\gamma}} = 1$. Thus $\{\eta_{\alpha\beta\gamma}\} \in \check{Z}^2(X;\mathbb{Z})$.
- The cohomology class $c_1([P]) \in \check{H}^2(X;\mathbb{Z})$ is represented by $\{\eta_{\alpha\beta\gamma}\}$.

It's tedious but routine to verify that the result is independent of choices.

Extension of structure group

Recall that a central extension $\tilde{G}$ of G is a short exact sequence of the form

$$0 \to A \to \tilde{G} \to G \to 0,$$

where A is an (abelian) subgroup of $\tilde{G}$. We want to know when it's possible to lift transition functions from a structure group to an extension. The prototypical example is

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(k) \to \mathrm{SO}(k) \to 0.$$

The group $\mathrm{Spin}(k)$ for $k > 1$ is characterized as the unique nontrivial $\mathbb{Z}_2$ extension of $\mathrm{SO}(k)$: Thus $\mathrm{Spin}(k)$ is the total space of a principal $\mathbb{Z}_2$-bundle over $\mathrm{SO}(k)$. These are classified topologically by

$$H^1(\mathrm{SO}(k);\mathbb{Z}_2) = \mathrm{Hom}(H_1(\mathrm{SO}(k);\mathbb{Z});\mathbb{Z}_2) \oplus 0 = \mathrm{Hom}(\pi_1(\mathrm{SO}(k))^{\mathrm{ab}};\mathbb{Z}_2).$$

We know that $\pi_1(\mathrm{SO}(2)) = \pi_1(S^1) = \mathbb{Z}$, and $\pi_1(\mathrm{SO}(3)) = \pi_1(S^3/\mathbb{Z}_2) = \mathbb{Z}_2$. It's easy to show that $\pi_1(\mathrm{SO}(k+1)) = \pi_1(\mathrm{SO}(k))$ for $k \geq 3$. Thus $H^1(\mathrm{SO}(k);\mathbb{Z}_2) = \mathbb{Z}_2$ has a unique nontrivial element corresponding topologically to $\mathrm{Spin}(k)$.

The part of the long exact sequence of Čech cohomology which makes sense is given by

$$H^1(X;\mathbb{Z}_2) \rightsquigarrow \check{H}^1(X;\mathrm{Spin}(k)_{C^\infty}) \to \check{H}^1(X;\mathrm{SO}(k)_{C^\infty}) \xrightarrow{w_2} H^2(X;\mathbb{Z}_2).$$

An isomorphism class $[P] \in \check{H}^1(X;\mathrm{SO}(k)_{C^\infty})$ comes from an element of $\check{H}^1(X;\mathrm{Spin}(k)_{C^\infty})$ if and only if $w_2([P]) = 0 \in H^2(X;\mathbb{Z}_2)$. There can be several isomorphism classes of principal $\mathrm{Spin}(k)$

bundles lifting the same class of $\mathrm{SO}(k)$ bundles. The action of $H^1(X;\mathbb{Z}_2)$ is transitive, but not always free. However this action becomes free if we refine our notion of lift. This refinement is an essential subtlety for the definition of a spin structure.

As before, suppose $0 \to A \to \tilde{G} \to G \to 0$ is a central extension, and P is some fixed principal G-bundle. A lift of P to the structure group $\tilde{G}$ is a principal $\tilde{G}$-bundle $\tilde{P}$ equipped with an isomorphism of P with the G-bundle associated to the quotient of $\tilde{P}$ by A. Two lifts are equivalent if they are related by an isomorphism of $\tilde{P}$ which induces *the identity* on P. (A general isomorphism of $\tilde{P}$ induces an isomorphism on P which is not necessarily the identity!) Be warned that it is possible for inequivalent lifts to be isomorphic as principal $\tilde{G}$-bundles.

To understand the equivalence classes of lifts of such a bundle P, suppose that $\{\phi_{\alpha\beta}\} \in \check{Z}^1(X;G)$ is a Čech cocycle representing the the transition functions for P relative to some local trivialization. If the cochain $\{\tilde{\phi}_{\alpha\beta}\} \in \check{Z}^1(X;\tilde{G})$ is an arbitrary choices for lifts to $\tilde{G}$, then the combination $w_2 := [\{\tilde{\phi}_{\alpha\beta}\tilde{\phi}_{\beta\gamma}\tilde{\phi}_{\gamma\alpha}\}] \in \check{H}^2(X;A)$ is called the generalized second Stiefel-Whitney class, and depends only on the isomorphism class of P, i.e. $[\{\phi_{\alpha\beta}\}] \in \check{H}^1(X;G)$. The cochain $\{\tilde{\phi}_{\alpha\beta}\}$ can be chosen to be a cocycle if and only if $w_2([P]) = 0$. Such a cochain then corresponds to a lift $\tilde{P}$ of P. Any other lift is of the form $\{a_{\alpha\beta}\tilde{\phi}_{\alpha\beta}\}$ for $\{a_{\alpha\beta}\} \in \check{Z}^1(X;A)$, and two such lifts are isomorphic if and only if they represent the same element of $\check{H}^1(X;A)$. In this manner, the space of lifts of P up to equivalence is an $\check{H}^1(X;A)$-torsor when $w_2([P]) = 0$, and empty otherwise.