Frame bundles

Given a smooth Euclidean vector bundle $\pi : E \to X$, last time we saw how to describe it using transition functions relative to a local trivialization $\{\phi_\alpha\}$.

$$\begin{array}{ccc} E & \xleftarrow[\{\phi_\alpha\}]{\cong} & \coprod_\alpha U_\alpha \times \mathbb{R}^k / \sim . \\ & {}_{\pi}\searrow \quad \swarrow_{\pi_1} & \\ & X & \end{array}$$

The transition functions are $\phi_{\alpha\beta} := \phi_\alpha^{-1} \circ \phi_\beta \in C^\infty(U_{\alpha\beta}; \mathrm{O}(k))$, and the equivalence relation is $[x, v]_\beta \sim [x, \phi_{\alpha\beta}(x)v]_\alpha$. We will see that the bundle of orthonormal frames $\mathrm{Fr}^{\mathrm{O}}(E)$ is a "principal bundle" which effectively encodes the transition functions geometrically, and thus is a more fundamental object.

For simplicity, we first study the frames of a single vector space. Let V be a Euclidean vector space of rank k. The *orthonormal frames* on V are the isometries

$$\mathrm{Fr}^{\mathrm{O}}(V) := \mathrm{Isom}(V \leftarrow \mathbb{R}^k).$$

Note that the image of the standard basis $\{e_i\}_{i=1}^k$ of $\mathbb{R}^k$ determines an orthonormal basis of V.

There is a natural right action of $\mathrm{O}(k) := \mathrm{Isom}(\mathbb{R}^k \leftarrow \mathbb{R}^k)$ on $\mathrm{Fr}^{\mathrm{O}}(V)$ by composition. Specifically, if $\phi \in \mathrm{Fr}^{\mathrm{O}}(V)$ and $g \in \mathrm{O}(k)$, then

$$\phi g \in \mathrm{Isom}(V \leftarrow \mathbb{R}^k \leftarrow \mathbb{R}^k) = \mathrm{Fr}^{\mathrm{O}}(V).$$

For any $\phi_0, \phi_1 \in \mathrm{Fr}^{\mathrm{O}}(V)$, there is a unique $g \in \mathrm{O}(k)$ such that $\phi_1 = \phi_0 g$, namely

$$\phi_1 = \phi_0 \underbrace{\left(\phi_0^{-1}\phi_1\right)}_{\in \mathrm{O}(k)}.$$

Thus if we fix ϕ_0, we see that $\mathrm{Fr}^{\mathrm{O}}(V)$ is in bijection with $\mathrm{O}(k)$ via $\phi \mapsto \phi_0^{-1}\phi$. But apart from such an identification, it makes no sense to compose elements of $\mathrm{Fr}^{\mathrm{O}}(V)$, so it is not a group. Instead, we say that $\mathrm{Fr}^{\mathrm{O}}(V)$ is a *right* $\mathrm{O}(k)$ *torsor*, where a torsor is a space with a free and transitive group action. More concretely, a torsor is a copy of a group that has lost its identity, but still knows how to act on itself by left or right multiplication.

Simultaneously, $\mathrm{Fr}^{\mathrm{O}}(V)$ is a left $\mathrm{Aut}(V) = \mathrm{O}(V) := \mathrm{Isom}(V \leftarrow V)$ torsor via composition on the other side:

$$\mathrm{O}(V) \circlearrowright \mathrm{Fr}^{\mathrm{O}}(V) \circlearrowleft \mathrm{O}(k),$$

and these actions clearly commute.

Although $\mathbb{R}^k$ and V are not naturally isomorphic, the two corresponding trivial bundles over $\mathrm{Fr}^{\mathrm{O}}(V)$ are. There is a natural isomorphism

$$\mathrm{Fr}^{\mathrm{O}}(V) \times \mathbb{R}^k \longrightarrow \mathrm{Fr}^{\mathrm{O}}(V) \times V$$

given by

$$(\phi, x) \mapsto (\phi, \phi(x)).$$

This can be turned into an important construction of V as follows. There is a diagonal left action of $\mathrm{O}(k)$ on $\mathrm{Fr}^{\mathrm{O}}(V) \times \mathbb{R}^k$ given by

$$g(\phi, x) = (\phi g^{-1}, gx).$$

The map

$$\begin{aligned} \mathrm{Fr}^{\mathrm{O}}(V) \times \mathbb{R}^k &\longrightarrow V \\ (\phi, x) &\mapsto \phi(x) \end{aligned}$$

is invariant under the action of $\mathrm{O}(k)$:

$$g(\phi, x) = (\phi g^{-1}, gx) \mapsto \phi(g^{-1}gx) = \phi(x).$$

It is straightforward to verify that we get an isomorphism

$$\frac{\mathrm{Fr}^{\mathrm{O}}(V) \times \mathbb{R}^k}{G} \xrightarrow{\cong} V.$$

To verify, fix any $\phi_0 \in \mathrm{Fr}^{\mathrm{O}}(V)$. On the quotient, note that equivalence is $[\phi g, x] \sim [\phi, gx]$. Then

$$[\phi, x] = \left[\phi_0 \phi_0^{-1} \phi, x\right] = \left[\phi_0, \phi_0^{-1} \phi x\right].$$

Thus every point is equivalent to a (unique) point of the form $[\phi_0, x']$ for $x' \in \mathbb{R}^n$, and clearly these representatives are mapped isomorphically via ϕ_0 to V.

There is an important generalization of this construction called an *associated space*. Suppose F is a space with an action of $\mathrm{O}(k)$, i.e. we have a homomorphism $\rho : \mathrm{O}(k) \to \mathrm{Aut}(F)$. We combine the right action on $\mathrm{Fr}^{\mathrm{O}}(V)$ with the left action of F to define

$$\mathrm{Fr}^{\mathrm{O}}(V) \times_\rho F := \frac{\mathrm{Fr}^{\mathrm{O}}(V) \times F}{G},$$

where equivalence is given by $[\phi g, f] \sim [\phi, \rho(g) f]$. This allows us to associate structures from the model space $\mathbb{R}^k$ to an abstract copy V, so long as the structure is $\mathrm{O}(k)$-invariant. For example, when ρ_{st} is the standard representation on $\mathbb{R}^k$, from the previous computation we get

$$\mathrm{Fr}^{\mathrm{O}}(V) \times_{\rho_{\mathrm{st}}} \mathbb{R}^k = V.$$

If ρ_{Λ^p} is the representation $\mathrm{O}(k) \to \mathrm{O}(\Lambda^p \mathbb{R}^k)$ on the p-th exterior power, then

$$\mathrm{Fr}^{\mathrm{O}}(V) \times_{\rho_{\Lambda^p}} \Lambda^p \mathbb{R}^k = \Lambda^p V.$$

If $\mathrm{Ad} : \mathrm{O}(k) \to \mathrm{Aut}(\mathrm{O}(k))$ is the adjoint action $g \mapsto (h \mapsto ghg^{-1})$, then

$$\mathrm{Fr}^{\mathrm{O}}(V) \times_{\mathrm{Ad}} \mathrm{O}(k) \cong \mathrm{O}(V)$$

via

$$[\phi, h] \mapsto \phi h \phi^{-1} \in \mathrm{Isom}(V \leftarrow \mathbb{R}^k \leftarrow \mathbb{R}^k \leftarrow V).$$

The adjoint action is appropriate since $[\phi g, h]$ and $[\phi, ghg^{-1}]$ correspond to the same element of $\mathrm{O}(V)$.

We can repeat these constructions fiberwise for a smooth Euclidean vector bundle of rank k, $\pi : E \to X$. The orthonormal frame bundle $\mathrm{Fr}^{\mathrm{O}}(E)$ is the fiber bundle over X whose fiber at a point x is $\mathrm{Fr}^{\mathrm{O}}(E|_x)$. It carries commuting actions

$$(\mathcal{G}_E := \Gamma(\mathrm{Aut}(E))) \circlearrowright \mathrm{Fr}^{\mathrm{O}}(E) \circlearrowleft C^\infty(X; \mathrm{O}(k)),$$

where $\Gamma(X; \mathrm{Aut}(E))$ denotes smooth sections of the bundle whose fiber over any point x are the isometries of $E|_x$. These are also known as *gauge transformations*.

Given $\rho : \mathrm{O}(k) \to \mathrm{Aut}(F)$, we can form the associated bundle

$$\mathrm{Fr}^{\mathrm{O}}(E) \times_\rho F := \frac{\mathrm{Fr}^{\mathrm{O}}(E) \times F}{\mathrm{O}(k)}$$

with the same equivalence relation fiberwise.

Given a local trivialization $\{\phi_\alpha\}$, the any associated bundle may be reconstructed via transition functions:

$$\begin{array}{ccc} \mathrm{Fr}^{\mathrm{O}}(E) \times_\rho F & \xleftarrow[\{\phi_\alpha\}]{\cong} & \coprod_\alpha U_\alpha \times F/\sim, \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & X & \end{array}$$

where $\phi_\alpha\left([x, f]_\alpha\right) = [\phi_\alpha(x), f]$. We compute that the necessary equivalence relation on $\coprod_\alpha U_\alpha \times F/\sim$ must be $[x, f]_\beta \sim \left[x, \rho(\phi_{\alpha\beta}) f\right]_\alpha$ by equating equivalent points in the image:

$$\begin{aligned} & [x, f']_\alpha \sim [x, f]_\beta \\ \iff & \phi_\alpha\left([x, f']_\alpha\right) \sim \phi_\beta\left([x, f]_\beta\right) \\ \iff & [\phi_\alpha(x), f'] \sim \left[\phi_\beta(x), f\right] \\ \iff & [\phi_\alpha(x), f'] \sim \left[\phi_\alpha(x)\phi_\alpha^{-1}(x)\phi_\beta(x), f\right] \\ \iff & [\phi_\alpha(x), f'] \sim \left[\phi_\alpha(x), \rho\left(\phi_{\alpha\beta}\right) f\right] \\ \iff & f' = \rho\left(\phi_{\alpha\beta}\right) f. \end{aligned}$$

Thus the associated bundle uses the same transition functions, but they are represented on a different fiber.

One important example is that the bundle $\mathrm{Aut}(E) := \mathrm{Fr}^{\mathrm{O}}(E) \times_{\mathrm{Ad}} \mathrm{O}(k)$, whose whose fiber over a point x is $\mathrm{O}(E|_x)$, and whose global sections are $\mathcal{G}_E := \Gamma(\mathrm{Aut}(E))$.

A *principal G-bundle* is a fiber bundle associated with the action $\rho_L : G \to \mathrm{Aut}(G)$ given by left multiplication $g \mapsto (h \mapsto gh)$. For example,

$$\mathrm{Fr}^{\mathrm{O}}(E) \times_{\rho_L} \mathrm{O}(k) \cong \mathrm{Fr}^{\mathrm{O}}(E)$$

by the map

$$[\phi(x), g] \mapsto \phi(x) g.$$

Thus $\mathrm{Fr}^{\mathrm{O}}(E)$ is a principal $\mathrm{O}(k)$ bundle.

More generally, given any fiber bundle $\pi : H \to X$ with fiber F, there is a principal $\mathrm{Aut}(F)$ bundle P such that the fiber $P|_x$ is $\mathrm{Iso}(H|_x \leftarrow F)$. There is clearly a right action on P by $\mathrm{Aut}(F)$ (which generalizes to an action of $C^\infty(X; \mathrm{Aut}(F))$). If ρ is the standard representation $\rho : \mathrm{Aut}(F) \to \mathrm{Aut}(F)$, then $P \times_\rho F = H$. In the case when $F = \mathbb{R}^k$ with the standard Euclidean structure, then H is a Euclidean vector bundle, and P is the orthonormal frame bundle.

The moral is that any fiber bundle H with fiber F is equivalent to a pair (P, ρ) where P is a principal G-bundle, and $\rho : G \to \mathrm{Aut}(F)$.

Čech cohomology revisited

Recall that Čech cohomology is described by

$$\check{C}^p(\{U_\alpha\}; A) := \left\{\phi = \left\{\phi_{\alpha_0\alpha_1\cdots\alpha_p} : U_{\alpha_0\alpha_1\cdots\alpha_p} \to A \text{ locally constant}\right\}\right\},$$

with differential

$$\begin{aligned} d : \check{C}^p(\{U_\alpha\}; A) &\to \check{C}^{p+1}(\{U_\alpha\}; A) \\ (d\phi)_{\alpha_0\alpha_1\cdots\alpha_{p+1}} &:= \sum_{k=0}^{p+1} (-1)^k \, \phi_{\alpha_0\cdots\widehat{\alpha_k}\cdots\alpha_{p+1}}. \end{aligned}$$

There is no reason to restrict to locally constant functions valued in an abelian group. In the context of principal bundles, we consider more general sheaves (i.e. classes of functions or sections) and try to make sense of Čech cohomology.

Recall that a local trivialization for a Euclidean vector bundle is a smooth map which for each $x \in U_\alpha$ gives an isometry $\phi_\alpha(x) : \mathbb{R}^k \to E|_x$. Each $\phi_\alpha(x)$ is an orthonormal frame, so it is equivalent to say that $\phi_\alpha \in \Gamma(U_\alpha; P)$ is a smooth section of the orthonormal frame bundle.

For a general principal bundle P, a local trivialization is equivalent to a local section. A system of local trivializations covering P is equivalent to a collection of local sections $\{\phi_\alpha \in \Gamma(U_\alpha; P)\}$. In the Čech framework,

$$\phi = \{\phi_\alpha\} \in \check{C}^0(X; P).$$

Transition functions are $\phi_{\alpha\beta} := \phi_\alpha^{-1}\phi_\beta \in C^\infty(U_{\alpha\beta}; G)$. This is the nonabelian version of an alternating sum with omitted indices, so we should interpret

$$d\{\phi_\alpha\} := \left\{\phi_{\alpha\beta} := \phi_\alpha^{-1}\phi_\beta\right\} \in \check{C}^1(X; C^\infty(G)).$$

The relation $d^2 = 0$ still holds if we interpret

$$d\left\{\phi_{\alpha\beta}\right\} := \left\{\phi_{\alpha\beta\gamma} := \phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta}\right\}.$$

The condition

$$\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta} = 1$$

is the *cocycle condition* for transition functions, and incorporates all the constraints for general transition functions. Usually the constraints are written with three formulas, but this form encodes them all into one. Setting $\alpha = \beta = \gamma$, we get

$$1 = \phi_{\alpha\alpha}\phi_{\alpha\alpha}^{-1}\phi_{\alpha\alpha} = \phi_{\alpha\alpha}.$$

Setting only $\gamma = \alpha$, we get

$$\phi_{\beta\alpha}\phi_{\alpha\alpha}^{-1}\phi_{\alpha\beta} = 1 \implies \phi_{\beta\alpha} = \phi_{\alpha\beta}^{-1}.$$

Finally,

$$\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta} = 1 \iff \phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1.$$