Problem 1. After proving the Hodge decomposition

$$\Omega^p(X) = (\text{image } \Delta) \oplus (\ker \Delta) = (\text{image } d) \oplus (\text{image } d^*) \oplus \mathscr{H}^p(X),$$

we defined the Green's operator

$$G : \Omega^p(X) \to \left(\mathscr{H}^p(X)^\perp \subset \Omega^p(X)\right)$$
$$G(\alpha) := \omega, \text{ where } \omega \in \mathscr{H}^p(X)^\perp \text{is the unique solution to } \Delta\omega = \alpha - \pi_{\mathscr{H}^p}(\alpha).$$

First verify that in the Hodge decomposition, image d, image d^*, and $\mathscr{H}^p(X)$ are mutually orthogonal. Next verify the claimed properties of G by showing that it

- is well-defined (i.e. $\omega \in \mathscr{H}^p(X)^\perp$ is *uniquely* specified by the defining equation).
- is linear.
- $\ker G = \mathscr{H}^p(X)$.
- is surjective, i.e. image $G = \mathscr{H}^p(X)^\perp$.
- commutes with d, d^*, and Δ. (Hint: first show that d and d^* commute with Δ).
- satisfies

$$\Delta G = G\Delta = \mathrm{Id}_{\Omega^p} - \pi_{\mathscr{H}^p}.$$

- is self-adjoint (Hint: show that $\langle\langle G(\alpha), \beta\rangle\rangle = \langle\langle \alpha, G(\beta)\rangle\rangle$ by using the Hodge decomposition $\beta = \Delta\eta + h$ where h is harmonic, and η is modified to be in $\mathscr{H}^p(X)^\perp$).
- is L^2-bounded, i.e. $\|G(\alpha)\|_{s+d} \leq C_s \|\alpha\|_s$ for some constant C_s (Hint: use the Poincaré inequality. It will also be useful to show that $\|\pi_{\mathscr{H}^p}(\alpha)\|_s \leq C'_s \|\alpha\|_s$, where $C'_s = \sum_{i=1}^r \|k_i\|_s \|k_i\|_{-s}$ for any orthonormal basis $\{k_i\}_{i=1}^r$ of $\mathscr{H}^p$.) Consequently, G extends to a map $L^2_s(X; \Lambda^p(T^*X)) \to L^2_{s+d}(X; \Lambda^p(T^*X))$.
- when viewed as a map $L^2_s(X; \Lambda^p(T^*X)) \to L^2_s(X; \Lambda^p(T^*X))$, G is a compact operator.

Problem 2. Let X be a closed oriented Riemannian n-manifold. For any fixed $\omega \in \Omega^p(X)$, consider the problem of finding an element $\omega + d\eta$ which minimizes the action

$$S(\omega + d\eta) := \tfrac{1}{2}\|\omega + d\eta\|^2_{L^2} = \tfrac{1}{2}\int_X |\omega + d\eta|^2 \underbrace{\text{dvol}}_{\star 1} = \tfrac{1}{2}\int_X (\omega + d\eta) \wedge \star(\omega + d\eta).$$

If $\omega_0 = \omega + d\eta$ is such an element, then the variational derivative must vanish:

$$\left.\frac{d}{dt}\right|_{t=0} S(\omega_0 + d(t\eta)) = 0 \;\; \forall \eta \in \Omega^{p-1}(X).$$

Using the vanishing of this variational derivative, determine the corresponding Euler-Lagrange equation for ω_0. Generalize to the case where X is compact, oriented with boundary.

Hint To deal with the case where X has a boundary, you will use the formula

$$\int_{\partial X} \alpha \wedge \star\beta = \langle\langle d\alpha, \beta\rangle\rangle_{L^2} - \langle\langle \alpha, d^*\beta\rangle\rangle_{L^2} .$$

First consider η supported on the interior of X. This will determine the equation satisfied by ω_0 on the interior of X. With this equation in hand, now consider general $\eta \in \Omega^{p-1}(X)$ to obtain the boundary condition.

Elliptic regularity

Let X be a closed oriented Riemannian manifold of dimension n, $E_i \to X$ Euclidean vector bundles for $i \in \{1, 2\}$, and $D : \Gamma(E_1) \to \Gamma(E_2)$ an elliptic (linear) differential operator of order d.

Exercise 3. Give a brief but careful proof that if β is any distributional section of E, and if $D\beta \in \Gamma(E_2)$, i.e. if $D\beta$ is a smooth section, then $\beta \in \Gamma(E_1)$, i.e. then β is smooth. Observe that since the zero section $0 \in \Gamma(E_2)$ is smooth, the subspace $\ker D$ does not depend on the choice of Sobolev completion

$$D : H^{s+d}(E_1) \to H^s(E_2),$$

i.e. although we enlarge the $\Gamma(E_i)$ to Sobolev completions, the subspace $\ker D$ does not shrink or grow.

Hint Use the elliptic estimate

$$\|\beta\|_{s+d} \le C_{D,s} \left(\|D\beta\|_s + \|\beta\|_s\right)$$

in an inductive argument to show that the Sobolev degree s of β can be made arbitrarily large. (This argument is called "elliptic bootstrapping." When D is nonlinear, the procedure is similar, but when s is small, the estimate is more delicate due to correction terms.)

Poincaré inequality

Exercise 4. For any $s \in \mathbb{R}$, show that there is a constant $C'_{D,s}$ such that there is an elliptic estimate of the form

$$\|\beta\|_{s+d} \le C'_{D,s} \|D\beta\|_s ,$$

for all $\beta \in L^2_{s+d}(X; E_1)$ which are L^2-orthogonal to $\ker D$.

Hint Fill in the details of the following argument.

First of all, the L^2 pairing between L^2_{s+d} and $\ker D$ is well-defined and finite, so the condition that $\beta \perp_{L^2} \ker D$ makes sense. Now we argue by contradiction. Suppose that no such constant $C'_{D,s}$ exists. Then there is a sequence $\{\beta_i\}$ with $\beta_i \perp_{L^2} \ker D$ such that the ratio $\|\beta_i\|_{s+d} / \|D\beta_i\|_s \to \infty$. Since this ratio is homogeneous, we can renormalize so that $\|\beta_i\|_{s+d} = 1$. After passing to a subsequence, we may assume that $\beta_i \overset{L^2_s}{\to} \beta$ for some $\beta \in L^2_s(X; E_1)$. Furthermore, $\beta \perp_{L^2} \ker D$. Finally, we compute $\|D\beta\|_{H^{s-d}} = 0$. This leads to an obvious contradiction.

Hint' Recall that the L^2 pairing extends to a perfect pairing between L^2_s and L^2_{-s}, so that $\alpha \cdot \beta \leq \|\alpha\|_s \|\beta\|_{-s}$. The kernel of any continuous linear map is a *closed* subspace (in particular, it is sequentially closed). Continuous maps of Banach spaces map take convergent sequences to convergent sequences, and limits to limits. The embeddings $L^2_{s+d} \hookrightarrow L^2_s$ and $L^2_s \hookrightarrow L^2_{s-d}$ are continuous and compact. To show that two distributions are equal, it suffices to show that their difference is zero under *any* norm.

Coarse Laplacian

Exercise 5. .

- Let V and W be Euclidean vector spaces. For $v \in V$, show that the adjoint of the map

$$T_v : W \to V \otimes W$$
$$T_v(w) := v \otimes w$$

  is determined by

$$T_v^* : V \otimes W \to W$$
$$T_v^*(v' \otimes w') = \langle v, v' \rangle \, w'.$$

- If $E \to X$ is a Euclidean vector bundle equipped with a connection A, then $\nabla_A : \Gamma(E) \to \Gamma(T^*X \otimes E)$. Compute the principal symbol $\sigma(\nabla_A) : T^*X \to \mathrm{Hom}(E, T^*X \otimes E)$. Deduce the principal symbols of $\nabla_A^* : \Gamma(T^*X \otimes E) \to \Gamma(E)$ and the "coarse Laplacian" $\nabla_A^* \nabla_A : \Gamma(E) \to \Gamma(E)$.

  Remark. Recall that the principal symbol of the Hodge Laplacian $\Delta : \Omega^k(X) \to \Omega^k(X)$ given by $\Delta = dd^* + d^*d$ is $\sigma(\Delta, x, p) = -|p|^2 \, \mathrm{Id}_{\Lambda^k}$. Taking the coarse Laplacian with $E = \Lambda^k T^*X$ and $A = \mathrm{LC}$ the Levi-Civita connection, we obtain a different second-order differential operator

$$\nabla_{\mathrm{LC}}^* \nabla_{\mathrm{LC}} : \Omega^k(X) \to \Omega^k(X).$$

- Based on the principal symbols, what can we conclude about the order of the differential operator given by the difference

$$\nabla_{\mathrm{LC}}^* \nabla_{\mathrm{LC}} - \Delta : \Omega^k(X) \to \Omega^k(X)?$$

Hint Recall that in a local trivialization,

$$\nabla_A(s^j e_j) = (\nabla + A) s^j e_j = (\partial_i s^j + A^j_{ik} s^k) \, dx^i \otimes e_j.$$

To compute the principal symbol $\sigma(\nabla_A, p, x)$, ignore all but the terms with the highest order derivatives, and replace ∂_i with the coordinate function p_i on the cotangent bundle. The principal symbol of a formal adjoint is given by $\sigma(D^*) = (-1)^d \sigma(D)^*$ where d is the degree of D. The symbol of a composition is the composition of symbols.

Note On $\Omega^0(X)$, we have $\Delta = \cancel{dd^*} + d^*d$, and we can identify ∇_{LC} with $d : \Omega^0(X) \to \Omega^1(X)$ and ∇_{LC}^* with $d^* : \Omega^1(X) \to \Omega^0(X)$. Thus $\nabla_{\mathrm{LC}}^* \nabla_{\mathrm{LC}} - \Delta = 0$. However, for $k > 0$ this difference is an operator involving the Riemannian curvature of X, given by the "Weitzenböck formula." This formula is a central lemma for compactness of the Seiberg-Witten moduli space.

Exercise 6. Suppose that $1 < p < \infty$ and $k \in \mathbb{Z}$ with $k \geq 1$ are such that $(k+1)/n - 1/p \geq 0$. (In particular, if $n = 4$ and $k = 1$, then $p \geq 2$.) Show that if $g \in \mathscr{G}_{k+1}^p$ and $A \in \mathscr{A}_k^p$, then

- $g \cdot A \in \mathscr{A}_k^p$,
- $F_A \in L_{k-1}^p(X; \Lambda^2 T^* X)$.

Hint Recall the local expressions

$$(g \cdot A)_\alpha = g_\alpha A_\alpha g_\alpha^{-1} - (dg_\alpha) g_\alpha^{-1},$$
$$(F_A)_\alpha = dA_\alpha + \tfrac{1}{2} \left[A_\alpha \wedge A_\alpha \right].$$

For the gauge transformation, note that L_k^p is a module for L_{k+1}^p. (The borderline case requires the additional hypothesis that $g \in L^\infty$, which follows from the compactness of G.) For the curvature, show that if L_k^p is below the borderline, then the multiplication is continuous. If L_k^p is above the borderline, then the multiplication is obviously continuous. If L_k^p is borderline, then we can escape below the borderline via the embedding $L_k^p \hookrightarrow L_k^{p'}$ for any p' satisfying

$$1/p < 1/p' \leq \min \left(1, 1/p + \left((k+1)/n - 1/p\right)/2\right).$$

Then we obtain continuous multiplication

$$L_k^p \times L_k^p \hookrightarrow L_k^{p'} \times L_k^{p'} \to L_{k-1}^p.$$